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Full Length Article

Exact errors of best approximation for complex-valued nonperiodic functions

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Abstract

We extend Markov's and Nagy's theorems on best approximation by entire functions of exponential type in the $L_1(\mathbb{R})$ metric to some complex-valued integrable and locally integrable functions. We use these results for finding sharp constants of best approximation in $L_1(\mathbb{R})$ and $L_\infty(\mathbb{R})$ on some complex convolution classes. For classes of real-valued convolutions those constants were found by Akhiezer. As an example, we apply these results to the Schwarz-type kernel and to the corresponding convolution classes.

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1. Introduction

In this paper we extend some results on best approximation by entire functions of exponential type in the $L_1(\mathbb{R})$ and $L_\infty(\mathbb{R})$ metrics to complex-valued nonperiodic functions.

Notation and preliminaries. For a complex number z we define $\operatorname{sgn} z$ to be $z/|z|$ if $z \neq 0$ and $\operatorname{sgn} 0 = 0$. Let B_σ be the set of all univariate complex-valued entire functions of exponential type $\sigma > 0$, and let $L_p(E)$ be the space of all measurable functions $f : E \to \mathbb{C}$ defined on a

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measurable subset E of the real axis $\mathbb{R}$ with the finite norm

$$\|f\|_{L_p(E)} := \begin{cases} \left(\int_E |f(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in E} |f(x)|, & p = \infty. \end{cases}$$

Note that two functions from $L_p(E)$ are called equal if they are equal almost everywhere on E. We also set $\|f\|_p := \|f\|_{L_p(\mathbb{R})}$, $1 \le p \le \infty$.

Next, for any locally integrable function $f : \mathbb{R} \to \mathbb{C}$, we define the error of best approximation in $L_p(\mathbb{R})$ by

$$A_\sigma(f)_p := \inf_{g\in B_\sigma} \|f - g\|_p, \qquad 1 \le p \le \infty.$$

The following properties of $A_\sigma(f)_p$ for $f \in L_p(\mathbb{R})$, $1 \le p \le \infty$, are valid:

1. $A_\sigma(f)_p$ is a nonincreasing function of $\sigma \in (0, \infty)$, $1 \le p \le \infty$.
2. $A_\sigma(f)_p$ is a continuous function of $\sigma \in (0, \infty)$, $1 \le p < \infty$; in particular,

$$A_{\sigma-0}(f)_1 = A_\sigma(f)_1. \tag{1.1}$$

3. $A_\sigma(f)_\infty$ is a continuous function of $\sigma \in (0, \infty)$ from the right. However, $A_\sigma(f)_\infty$ can be discontinuous from the left. For instance if $f \in L_\infty(\mathbb{R})$ is a periodic function, then $A_{\sigma-0}(f)_\infty < A_\sigma(f)_\infty$ for some $\sigma > 0$.
4. If $f \in L_\infty(\mathbb{R})$ is $2\pi/\sigma$-periodic on $\mathbb{R}$, then

$$A_{\sigma-0}(f)_\infty = \inf_{c\in\mathbb{C}} \|f - c\|_\infty. \tag{1.2}$$

Properties 1–3 are proved in [1, Sect. 99] and [19, Sect. 2.6.22], while property 4 is established in [1, Sect. 96] and [19, Sect. 2.6.22].

We say that $g_\sigma \in B_\sigma$ is an entire function of exponential type of best approximation (EFBA) to a locally integrable function f in $L_p(\mathbb{R})$ if $A_\sigma(f)_p = \|f - g_\sigma\|_p$, $1 \le p \le \infty$. If $A_\sigma(f)_p < \infty$, then an EFBA to f in $L_p(\mathbb{R})$, $1 \le p \le \infty$, always exists (cf. [19, Sects. 2.6.2 and 2.6.3]).

The Fourier transform of a function $\varphi \in L_1(\mathbb{R})$ is defined by the formula $\mathcal{F}(\varphi)(x) := \int_{\mathbb{R}} \varphi(t)e^{ixt}dt$. We use the same notation $\mathcal{F}(\Phi)$ for the Fourier transform of a tempered distribution Φ on $\mathbb{R}$. By the definition (see for example [18, Sect. 4.1]), Φ is a continuous linear functional $\langle\Phi, h\rangle$ on the Schwartz class $S(\mathbb{R})$ of all test functions h on $\mathbb{R}$, and $\mathcal{F}(\Phi)$ is defined by the formula $\langle F(\Phi), h\rangle := \langle\Phi, F(h)\rangle$, $h \in S(\mathbb{R})$. In some cases discussed in the paper, functionals $\langle\Phi, \cdot\rangle$ or $\langle F(\Phi), \cdot\rangle$ are generated by integrable or locally integrable functions. In particular, if Φ is a tempered distribution on $\mathbb{R}$ whose restriction φ to $\mathbb{R} \setminus (-\sigma, \sigma)$ is an integrable function, then $\langle\Phi, h\rangle = \int_{\mathbb{R}} \varphi(t)h(t)dt$ for all $h \in S(\mathbb{R})$ with $\operatorname{supp} h \subseteq \mathbb{R} \setminus (-\sigma, \sigma)$. If $f = F(\Phi)$ is a locally integrable function on $\mathbb{R}$, then $\langle F(\Phi), h\rangle = \int_{\mathbb{R}} f(t)F(h)(t)dt$ for all $h \in S(\mathbb{R})$.

We also use generic notation $\Gamma(z)$ and $F(a, b; c; z)$ for the gamma function and the hypergeometric function, respectively.

L_1-approximation of real-valued functions. Such topics of approximation theory as finding exact errors of best approximation for individual functions and finding sharp constants of best approximation on functional classes have always attracted much attention of approximation analysts.

Due to a Markov-type theorem [13, Theorem 3, p. 205], [1, Sect. 50] and Nagy's results [14,15], exact errors of best L_1-approximation by trigonometric polynomials and entire functions