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Full Length Article

Schoenberg's theorem for real and complex Hilbert spheres revisited

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Abstract

Schoenberg's theorem for the complex Hilbert sphere proved by Christensen and Ressel in 1982 by Choquet theory is extended to the following result: Let L denote a locally compact group and let $\overline{\mathbb{D}}$ denote the closed unit disc in the complex plane. Continuous functions $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$ such that $f(\xi \cdot \eta, u^{-1}v)$ is a positive definite kernel on the product of the unit sphere in $\ell_2(\mathbb{C})$ and L are characterized as the functions with a uniformly convergent expansion

$$f(z, u) = \sum_{m,n=0}^{\infty} \varphi_{m,n}(u) z^m \overline{z}^n,$$

where $\varphi_{m,n}$ is a double sequence of continuous positive definite functions on L such that $\sum \varphi_{m,n}(e_L) < \infty$ (e_L is the neutral element of L). It is shown how the coefficient functions $\varphi_{m,n}$ are obtained as limits from

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expansions for positive definite functions on finite dimensional complex spheres via a Rodrigues formula for disc polynomials.

Similar results are obtained for the real Hilbert sphere.

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1. Introduction and main results

Characterizations of positive definite functions on spheres or on products of spheres with locally compact groups can be found in the literature, see [4–7,10,15,19]. The spheres can be real or complex and of finite or countably infinite dimension. In this paper we obtain a characterization of positive definite functions on the product of the unit sphere in the complex Hilbert space $\ell_2(\mathbb{C})$ with a locally compact group via a power series expansion. This is the only missing case in the above picture. We show how the coefficients of this expansion are related with those of the expansions on the product of finite dimensional complex spheres with locally compact groups.

In the case of real spheres there has been several statistical applications of such results, see [1,9,16], but we do not know of statistical applications in the case of complex spheres. The interested reader is referred to [13], where parametric families of positive definite functions on complex spheres are provided.

In order to arrive quickly at the main results of the paper, we postpone precise definitions to Section 2.

Schoenberg's theorems in [19] for real spheres $\mathbb{S}^d, d = 1, 2, \dots, \infty$, give uniformly convergent expansions

$$f(x) = \sum_{n=0}^{\infty} \varphi_{n,d} c_n(d, x), \quad x \in [-1, 1] \tag{1}$$

for certain classes $\mathcal{P}(\mathbb{S}^d)$ of continuous functions $f : [-1, 1] \to \mathbb{R}$. Here, $c_n(d, x)$ are normalized ultraspherical polynomials when $d \in \mathbb{N}$, see (13), while $c_n(\infty, x) = x^n$. Furthermore, $(\varphi_{n,d})_{n\geq 0}$ is a sequence of non-negative numbers satisfying $\sum_n \varphi_{n,d} < \infty$.

Schoenberg's theorems were extended in [5] to classes $\mathcal{P}(\mathbb{S}^d, L)$ of continuous functions $f : [-1, 1] \times L \to \mathbb{C}$, where L is an arbitrary locally compact group. In this case the uniformly convergent expansions are

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad x \in [-1, 1], u \in L, \tag{2}$$

and the expansion coefficients $\varphi_{n,d}$ from (2) belong to the class $\mathcal{P}(L)$ of continuous positive definite functions on the group L. They are called the d-Schoenberg functions associated with f. The extension was motivated by problems in geostatistics in the particular case, where L is the additive group $\mathbb{R}$ of real numbers representing time.

In the special case where L is reduced to the neutral element e_L, the results of [5] yield Schoenberg's theorems.