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Full Length Article

Random N-continued fraction expansions

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Abstract

The N-continued fraction expansion is a generalization of the regular continued fraction expansion, where the digits 1 in the numerators are replaced by the natural number N. Each real number has uncountably many expansions of this form. In this article we focus on the case $N = 2$, and we consider a random algorithm that generates all such expansions. This is done by viewing the random system as a dynamical system, and then using tools from ergodic theory to analyse these expansions. In particular, we use a recent Theorem of Inoue (2012) to prove the existence of an invariant measure of product type whose marginal in the second coordinate is absolutely continuous with respect to Lebesgue measure. Also some dynamical properties of the system are shown and the asymptotic behaviour of such expansions is investigated. Furthermore, we show that the theory can be extended to the random 3-continued fraction expansion.

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1. Introduction

In 2008 Burger et al. introduced in [4] the N-continued fraction expansion. Given an $x \in \mathbb{R}$ and an $N \geq 1$ they showed that x can be represented in the following way:

$$x = d_0 + \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots}}}, \tag{1}$$

where the digits $d_i \in \mathbb{N}$. Anselm and Weintraub showed in [2] that every $x \in \mathbb{R}$ has in fact infinitely many such expansions. Dajani et al. obtained in [7] the N-continued fraction expansions from transformations of the form

$$S_{N,i}(x) = \begin{cases} \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor + i & \text{if } x \in \left(0, \frac{N}{i+1}\right] \\ \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor & \text{if } x \in \left(\frac{N}{i+1}, N\right] \\ 0 & \text{if } x = 0 \end{cases}, \tag{2}$$

where $N \in \mathbb{N}$ and $i \in \{0, 1, \ldots, N-1\}$. Approaching the N-continued fraction expansions as a dynamical system Dajani et al. showed that the result obtained in [2] is immediate. They also gave invariant measures for several transformations generating N-expansions.

In this paper we will consider N-continued fraction expansions generated by a random dynamical system. A random dynamical system consists of a family of transformations on a state space and a probability distribution on the family of transformations. For each iterate a transformation of the family is chosen according to the probability distribution. In this paper we use the family of transformations $\{S_{N,i}, i \in \{0, 1, \ldots, N-1\}\}$ where $S_{N,i}$ are given by (2). The main question is whether we can find an invariant measure for this random system. The existence of invariant measures for random systems has been studied frequently over the past decades. We will use a recent theorem of Inoue, [9] to ensure the existence of an invariant measure.

Defining the random dynamical system as a skew product allows one to use results from ergodic theory in order to gain information about the asymptotic behaviour of the expansions. This is done in [10] for expansions like (1), where $N \in \{-1, 1\}$. In [8] more invariant measures for random β-expansions are obtained by constructing an isomorphism between the skew product for the random β-expansion and the digit sequences it induces. In this paper we will prove the existence of an invariant measure for the random transformation generating 2-continued fraction expansions, so expansions of the form (1) where $N = 2$. We will use the approach of [10] to show that an accelerated version of our system has an invariant measure of the form $m \times \mu$, where m is a Bernoulli measure and μ is equivalent with the Lebesgue measure. Using standard techniques, we lift the obtained invariant measure for the accelerated system to an invariant measure ρ for the original random system. We will write the random dynamical system as a skew product to obtain asymptotic properties of expansions like (1) using ergodic theoretic methods. Constructing an isomorphism between the skew product and the digit sequences obtained from the random dynamical system, allows us to show the existence of invariant measures which are singular with respect to the measure ρ mentioned above.

The paper is organized as follows. In Section 2 we define the random N-continued fraction transformation. In Section 3, we state the existence theorem of invariant measures for random transformations given by Inoue in [9], and show how we can apply this theorem to an induced transformation of the random 2-continued fraction transformation. In Section 4 we will define