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Full Length Article

Optimal order Jackson type inequality for scaled Shepard approximation

Steven Senger[a], Xingping Sun[a,*], Zongmin Wu[b]

[a] *Department of Mathematics, Missouri State University, Springfield, MO 65897, USA*

[b] *Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Science, Fudan University, Shanghai, China*

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Abstract

We study a variation of the Shepard (1968) approximation scheme by introducing a dilation factor into the base function, which synchronizes with the Hausdorff distance between the data set and the domain. The novelty enables us to establish an optimal order Jackson (1911) type error estimate (with an explicit constant) for bounded continuous functions on any given convex domain. We also improve en route an upper bound estimate due to Narcowich and Ward (1991) for the numbers of well-separated points in thin annuli, which is of independent interest.

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1. Introduction

When dealing with real world problems with high degrees of complexity, scientists often observe that unknown target functions are imprecise and elusive, and that data acquired on them

* Corresponding author.
E-mail address: xsun@missouristate.edu (X. Sun).

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do not always reflect their true nature. This can be caused by a host of known and unknown reasons. To name just a few, one may encounter reading and interpreting errors, lost in translation, instrument failures or malfunctions. In the literature, this is often referred to as the "noisy data" phenomenon; see [4]. Under these circumstances, models established by computationally-expensive algorithms often do not survive the test of cross validation and sometimes fail outright to reflect reality in ways they are designed for. Long and hard work devoted to the establishment of the models is quickly rendered worthless. To make meaningful decisions, one needs to use the best available data to build multiple models and test them against newly acquired data. The process is often repeated numerous times, as the so called "best available data" and model selection criteria may also subject to uncertainty of various degrees.

The challenging computing environment has ruled out the employability of most of the interpolation methods and many of the quasi-interpolation methods. An interpolation procedure entails solving a large linear system, which is often expensive and slow. In addition, interpolating "noisy data" is like playing a meaningless hide-and-seek game during which one seldom knows the boundary between "over-fitting" and "under-fitting". Many quasi-interpolation schemes rely on elaborated mathematical procedures to find the values of the parameters, which is unpractical in reality.

Shepard [20] proposed in $\mathbb{R}^2$ the following approximation scheme. Suppose function values $f(x_j)$ $(1 \leq j \leq n)$ are available at the scattered sites $\{x_1, \ldots, x_n\}$ in a domain X. Then a function of the form:

$$x \mapsto S_{\Phi,n}(x) := [\Phi_n(x)]^{-1} \sum_{j=1}^{n} f(x_j)\, \phi(|x - x_j|),$$

is constructed to approximate the target function f in X. Here $\phi : x \mapsto |x|^{-\lambda},\ \ \lambda \geq 1$, and $\Phi_n(x) = \sum_{j=1}^{n} \phi(x - x_j)$, in which $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^2$. This procedure has since become known as "Shepard approximation". This approximation scheme and others like it, such as combined Shepard's method, triangular Shepard's method, have been widely studied in [1,3,5,6,9,12,11,21,22,24–26], and the references therein.

In a nutshell, a Shepard approximation scheme employs rational formations of shifts of an appropriately-selected base function to approximate a target function, and its efficiency epitomizes in the reproduction of constants. Because of the singularity of the base function at zero, the original Shepard approximant interpolates values of the target function at $x_j,\ \ 1 \leq j \leq n$. That is, $S_{\Phi,n}(x_j) = f(x_j),\ \ 1 \leq j \leq n$. However, for most other choices of base functions, the interpolation feature is lost, and the resulted approximants are called "quasi-interpolants" in the literature; see [6,25,26]. Except for the cases in which the base function is compactly supported (see [24–26]), deriving optimal order error estimates for a Shepard approximation scheme has been uncommon. In this paper, we study a new variation of Shepard approximation by introducing a dilation factor into the base function, which synchronizes with the Hausdorff distance between the data set and the domain. The novelty enables us to establish an optimal order Jackson [15] type error estimate (with an explicit constant) for bounded continuous functions. The proof requires decomposing the domain as the union of concentric thin annuli with no common interior. This is a standard technique in many branches of analysis. Notably, the technique has recently been applied by authors of [14] and [13] in bounding the L_∞-norms of interpolation operators and the least square operators associated with radial basis functions. The following question arises naturally: how many well-separated points can be put inside a specified annulus? For the special case in which the annuli have thickness δ and inner radius $j\delta,\ \ j \in \mathbb{N}$, where δ is the separation radius of the data set, Narcowich and Ward [19] gave the upper bound