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Symmetric contours and convergent interpolation

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Abstract

The essence of Stahl–Gonchar–Rakhmanov theory of symmetric contours as applied to the multipoint Padé approximants is the fact that given a germ of an algebraic function and a sequence of rational interpolants with free poles of the germ, if there exists a contour that is "symmetric" with respect to the interpolation scheme, does not separate the plane, and in the complement of which the germ has a single-valued continuation with non-identically zero jump across the contour, then the interpolants converge to that continuation in logarithmic capacity in the complement of the contour. The existence of such a contour is not guaranteed. In this work we do construct a class of pairs interpolation scheme/symmetric contour with the help of hyperelliptic Riemann surfaces (following the ideas of Nuttall and Singh, 1977; Baratchart and Yattselev, 2009). We consider rational interpolants with free poles of Cauchy transforms of non-vanishing complex densities on such contours under mild smoothness assumptions on the density. We utilize $\bar{\partial}$-extension of the Riemann–Hilbert technique to obtain formulae of strong asymptotics for the error of interpolation.

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1. Introduction

Rational approximation of analytic functions is a very classical subject with various applications in number theory [23,36,37], numerical analysis [24,12], modeling and control of signals and systems [1,13,7,30], quantum mechanics and quantum field perturbation theory [5,44], and many others. The theoretical aspects of the theory include the very possibility of such an approximation [34,26,45] as well as the rates of convergence of the approximants at regular points when the degree grows large [46,20,29,31].

In this work we are interested in rational interpolants with free poles, the so-called *multipoint Padé approximants* [6]. Those are rational functions of type[1] (m, n) that interpolate a given function at $m + n + 1$ points, counting multiplicity. The beauty of multipoint Padé approximants lies in the simplicity of their construction and the connection to (non-Hermitian) orthogonal polynomials. More precisely, the approximated function always can be written as a Cauchy integral of a complex density on any curve separating the interpolation points from the singularities of the function. The denominators of the multipoint Padé approximants then turn out to be orthogonal to all the polynomials of smaller degree with respect to this density divided by the polynomial whose zeros are the finite interpolation points. This connection is the most fruitful when the curve can be collapsed into a contour that does not separate the plane (as in the case of functions with algebraic and logarithmic singularities only). In general, there are many choices for such a contour with no obvious geometrical reason to prefer one over the other. The identification of the "proper contour", the one that attracts almost all of the poles of the approximants, is a fundamental question in the theory of Padé approximation.

For the case of classical diagonal Padé approximants (all the interpolation points are at infinity and $m = n$) to functions with branchpoints this question was answered in a series of pathbreaking papers [38–40] by Stahl, where the approximants were shown to converge in capacity on the complement of the system of arcs of minimal logarithmic capacity outside of which the function is analytic and single-valued. The extremal system of arcs, called a *symmetric contour* or an *S-contour*, is characterized by the equality of the one-sided normal derivatives of its equilibrium potential at every smooth point of the contour, and the above-mentioned convergence ultimately depends on a deep potential-theoretic analysis of the zeros of non-Hermitian orthogonal polynomials. Shortly after, this result was extended by Gonchar and Rakhmanov [21] to multipoint Padé approximants to Cauchy integrals of continuous quasi-everywhere non-vanishing functions over contours minimizing now some *weighted* capacity, provided that the interpolation points asymptotically distribute like a measure whose potential is the logarithm of the weight, see Section 2 for a more detailed description of Stahl–Gonchar–Rakhmanov theory.

These works clearly show that the appropriate Cauchy integrals for Padé approximation must be taken over S-contours symmetric with respect to the considered interpolation schemes, if such contours exist. This poses a natural inverse problem: given a system of arcs, say Δ, is there an interpolation scheme turning Δ into a symmetric contour? This inverse problem was first considered by Baratchart and the author in [9] for the case of a single Jordan arc. Below we build on the ideas of [9] by exhibiting a class of contours that are symmetric with respect to appropriately constructed interpolation schemes, see Section 3.1, and then derive formulae of *strong* asymptotics for the error of approximation by multipoint Padé approximants to Cauchy integrals of smooth densities on these contours, see Section 3.3.

[1] A rational function is said to be of type (m, n) if it can be written as the ratio of a polynomial of degree at most m and a polynomial of degree at most n.