Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 225 (2018) 148–175

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

Properties of generalized Freud polynomials

Peter A. Clarkson[a,*], Kerstin Jordaan[b]

[a] School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, CT2 7FS, UK
[b] Department of Decision Sciences, University of South Africa, Pretoria, 0003, South Africa

Received 20 June 2016; received in revised form 20 July 2017; accepted 3 October 2017
Available online 19 October 2017

Communicated by Doron S. Lubinsky

Abstract

We consider the semiclassical generalized Freud weight function

$$w_\lambda(x;t) = |x|^{2\lambda+1}\exp(-x^4+tx^2), \qquad \lambda > -1, \quad x \in \mathbb{R}.$$

We analyse the asymptotic behaviour of the sequences of monic polynomials that are orthogonal with respect to $w_\lambda(x;t)$, as well as the asymptotic behaviour of the recurrence coefficient, when the degree, or alternatively, the parameter t, tend to infinity. We also investigate existence and uniqueness of positive solutions of the nonlinear discrete equation satisfied by the recurrence coefficients and prove properties of the zeros of the generalized Freud polynomials.

MSC: 33C47; 34M55; 65Q99

Keywords: Generalized Freud weight; Semiclassical orthogonal polynomials; Recurrence coefficients; Discrete Painlevé equation; Painlevé equation

1. Introduction

The study of polynomials orthogonal on unbounded intervals with respect to general exponential-type weights $\exp\{-Q(x)\}$, with $Q(x)$ a polynomial of the form $Q(x) = |x|^\alpha$, with $\alpha \in \mathbb{N}$, began with Géza Freud in the 1960's (for details see [23,24,54,55] as well as the

* Corresponding author.
E-mail addresses: P.A.Clarkson@kent.ac.uk (P.A. Clarkson), jordakh@unisa.ac.za (K. Jordaan).

https://doi.org/10.1016/j.jat.2017.10.001

monographs by Levin and Lubinsky [37] and Mhaskar [49]). Earlier Freud [24,25] investigated the asymptotic behaviour of the recurrence coefficients for special classes of weights by a technique giving rise to an infinite system of nonlinear equations called Freud equations for these coefficients, cf. [45,46]. If the monic orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfying the three-term recurrence relation

$$p_{n+1}(x) = xp_n(x) - \beta_n p_{n-1}(x), \tag{1.1}$$

with $p_{-1}(x) = 0$ and $p_0(x) = 1$, are related to the weight $w(x) = \exp(-x^4)$ on the whole real line, then the Freud equations are reduced to (cf. [5,25,36,44,52,53,55])

$$4\beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) = n, \tag{1.2a}$$

with initial conditions

$$\beta_0 = 0, \qquad \beta_1 = \frac{\int_{-\infty}^{\infty} x^2 \exp(-x^4)\,dx}{\int_{-\infty}^{\infty} \exp(-x^4)\,dx} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}. \tag{1.2b}$$

We remark that Eq. (1.2a) was first derived by Shohat [60, equation (39), p. 407]. Nevai [53] proved that there is a unique positive solution to the problem (1.2).

Freud [24], via the Freud equations, conjectured that the asymptotic behaviour of recurrence coefficients β_n in the recurrence relation (1.1) satisfied by the polynomials $\{p_n(x)\}_{n=0}^{\infty}$ orthogonal with respect to the weight

$$w(x) = |x|^{\lambda} \exp(-|x|^m), \qquad m \in \mathbb{N}, \tag{1.3}$$

with $\lambda > -1$, could be described by

$$\lim_{n\to\infty} \frac{\beta_n}{n^{2/m}} = \left[\frac{\Gamma(\frac{1}{2}m)\Gamma(1+\frac{1}{2}m)}{\Gamma(m+1)}\right]^{2/m}. \tag{1.4}$$

We note that Freud [24] proved the result for orthonormal polynomials whilst (1.4) is for monic orthogonal polynomials. Freud showed that if the limit exists for $m \in 2\mathbb{Z}$, then it is equal to the expression in (1.4) but could only prove the existence of the limit (1.4) for $m = 2, 4, 6$. Significant progress in the study of orthogonal polynomials associated with Freud weights was made when Magnus [44] proved the validity of Freud's conjecture for the recurrence coefficients when m is an even positive integer and weight

$$w(x) = \exp\{-Q(x)\}, \tag{1.5}$$

where $Q(x)$ is an even degree polynomial with positive leading coefficient. A more general proof of Freud's conjecture of the recursion coefficients for exponential weights is due to Lubinsky, Mhaskar, and Saff [43]; see also [15,23,24,55]. Deift et al. [16] discuss the asymptotics of orthogonal polynomials with respect to the weight (1.5) using a Riemann–Hilbert approach.

Bauldry, Máté, and Nevai [5] showed that the convergent solutions of a system of smooth recurrence equations, whose Jacobian matrix satisfies a certain non-unimodularity condition, can be approximated by asymptotic expansions and they provide an application to approximate the recurrence coefficients associated with polynomials orthogonal with respect to the weight (1.5), where $Q(x)$ is an even degree polynomial with positive leading coefficient. Further, Bauldry, Máté, Nevai and Zaslavsky obtained asymptotic expansions for the recurrence coefficients of a larger class of orthogonal polynomials with exponential-type weights, cf. [48, Theorem 1, p. 496] and [5, Theorem 5.1, p. 223].