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Greedy algorithm with gaps

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Abstract

We generalize the well-known greedy approximation algorithm, by allowing gaps in the approximating sequence. We give examples of bases which are "quasi-greedy with gaps," in spite of failing to be quasi-greedy in the usual sense. However, we also show that for some classical bases (such as the normalized Haar basis in L_1, and the trigonometric basis in L_p for $p \neq 2$), the greedy algorithm may diverge, even if gaps are introduced into the approximating sequence.

Keywords: Greedy algorithm; Basis

1. Introduction

We introduce and investigate a generalization of the well-known Thresholding Greedy Algorithm. First we need to fix some notation. If $(e_i, e_i^*)_{i \in I} \subset X \times X^*$ is a biorthogonal system, and A is a finite subset of $\mathbb{N}$, we write $Ax = \sum_{i \in A} \langle e_i^*, x \rangle e_i$, and $A^c x = x - Ax$. By default we assume that all our biorthogonal systems are infinite (that is, X is infinite dimensional), complete ($\overline{\text{span}[e_i : i \in I]} = X$) and total ($\overline{\text{span}[e_i^* : i \in I]}^{\text{weak}*} = X^*$), and further, $\sup_i \max\{\|e_i\|, \|e_i^*\|\} < \infty$ (the systems are *semi-normalized*). We would sometimes refer to $(e_i)_{i \in I}$ as a *basis*. The *support* of x (supp x) is the set $\{i : \langle e_i^*, x \rangle \neq 0\}$. By our assumption, elements with finite support are dense in X.

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We say that a set $A \subset I$ is *a t-greedy set for $x \in X$* (here $t \in (0, 1]$) if

$$\inf_{i \in A} \left|\langle e_i^*, x\rangle\right| \geq t \sup_{i \notin A} \left|\langle e_i^*, x\rangle\right|$$

(if x has infinite support, then such a set A is necessarily finite, since, for every $\varepsilon > 0$, the set $\{i : |\langle e_i^*, x\rangle| > \varepsilon\}$ is finite, see [5]). A *t-greedy projection of x* $\mathbf{G}_m^t(x)$ is Ax, where A is a t-greedy set of cardinality m (such a projection need not be unique). When $t = 1$, we simply talk about greedy sets, and greedy projections $\mathbf{G}_m$.

Suppose $\mathbf{n} = (n_1 < n_2 < \cdots)$ is an increasing sequence of positive integers, and $0 < t \leq 1$. The basis (e_i) is **n**-*quasi-greedy with parameter t* (**n**-*t-QG* for short) if $\lim_i \mathbf{G}_{n_i}^t x = x$ for any $x \in X$, and any choice of t-greedy projections $\mathbf{G}_{n_i}^t x$. When $t = 1$, we use the term **n**-*quasi-greedy* (**n**-*QG*). When $\mathbf{n} = \mathbb{N}$, we recover the usual notion of a basis being t-quasi-greedy, or quasi-greedy for $t = 1$ (we use the acronyms t-QG and QG).

This paper is structured as follows. In Section 2, we obtain a criterion for a basis being **n**-t-QG (similar to the criterion for the quasi-greediness of a basis from [18]). We also show that any basis which is **n**-QG with constant 1 must be 1-suppression unconditional. Examples of **n**-QG bases which are not QG are given in Section 3. In Section 4 we show that, if a set **n** is "large enough" (more precisely, if it "almost" forms an additive basis in $\mathbb{N}$), then any **n**-QG basis is automatically QG. In Section 5, we show that certain "classical" bases (the Haar basis in L_1, and the trigonometric basis in a weighted L_p space) are not **n**-QG, for any sequence **n**.

Throughout the paper, we use standard facts and notation from Banach spaces and approximation theory. We refer the reader to e.g. [15–17] for the necessary background. For a normed space X, we denote by $\mathbf{B}(X) = \{x \in X : \|x\| \leq 1\}$ its closed unit ball. We write "$x \precsim y$" ("$x \succsim y$", "$x \sim y$") as a shorthand for "there exists $c > 0$ so that $x/y \leq c$" (resp. $c^{-1} \leq x/y$, $c^{-1} \leq x/y \leq c$). For a collection of vectors $(e_i)_{i \in I}$, and a finite set $A \subset I$, we let $\mathbf{1}_A = \sum_{i \in A} e_i$. More generally, for $\varepsilon_i = \pm 1$, we set $\mathbf{1}_{\varepsilon A} = \sum_{i \in A} \varepsilon_i e_i$.

Remark 1.1. One can define **n**-greedy, and **n**-almost greedy bases, in the manner similar to the definition of **n**-QG bases. More precisely, suppose $(e_i, e_i^*)_{i \in I} \subset X \times X^*$ is a biorthogonal system. For $x \in X$, we can define

$$\sigma_m(x) = \inf_{|\mathrm{supp}\, y| \leq m} \|x - y\| \text{ and } \tilde{\sigma}_m(x) = \inf_{|A| \leq m} \|x - Ax\|.$$

The above basis is said to be **n**-*greedy* (resp. **n**-*almost greedy*) if there exists an absolute constant $C > 0$ so that, for any x, and any $n \in \mathbf{n}$, we have $\|x - \mathbf{G}_n x\| \leq C\sigma_n(x)$, resp. $\|x - \mathbf{G}_n x\| \leq C\tilde{\sigma}_n(x)$ (that is, the greedy approximation is "almost optimal"). However, such bases are necessarily greedy, resp. almost greedy, when I is infinite.

We consider the greedy case, as the almost greedy one is handled similarly. Suppose (e_i) is **n**-greedy, with constant C. We have to show that, for any x, and any greedy set A, we have $\|x - Ax\| \leq C\sigma_m(x)$, where $m = |A|$. For $\varepsilon > 0$ pick $y \in X$ so that $|\mathrm{supp}(y)| = m$ and $\|x - y\| < \sigma_m(x) + \varepsilon$.

Pick $n \in \mathbf{n} \cap [m, \infty)$. If $n = m$, we are done. Otherwise, by a "small perturbation argument", we can assume that x has finite support. Find a set B of cardinality $n - m$, disjoint from $\mathrm{supp}\, x \cup \mathrm{supp}\, y$. Set $x' = x + \sum_{k \in B} r e_i$, where $r > \max_i |\langle e_i^*, x\rangle|$. Then $A \cup B$ is a greedy set (of cardinality n) for x', and $x' - (A \cup B)x' = x - Ax$. However, $x' - (Bx' + y) = x - y$, hence $\|x - y\| \geq \sigma_n(x')$. Thus,

$$\|x - Ax\| = \|x' - (A \cup B)x'\| \leq C\sigma_n(x') \leq C\|x - y\| \leq C(\sigma_m(x) + \varepsilon).$$

As ε can be arbitrarily small, we are done.