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On nuclearity of embeddings between Besov spaces

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Abstract

Let $B^{s,\alpha}_{p,q}(\Omega)$ be the Besov space with classical smoothness s and additional logarithmic smoothness of order α on a bounded Lipschitz domain Ω in $\mathbb{R}^d$. For $s_1, s_2 \in \mathbb{R}$, $1 \le p_1, p_2, q_1, q_2 \le \infty$ and $s_1 - s_2 = d - d(1/p_2 - 1/p_1)_+$, we show a sufficient condition on q_1, q_2 for nuclearity of embedding $B^{s_1,\alpha_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2,\alpha_2}_{p_2,q_2}(\Omega)$. We also show that the condition is necessary in a wide range of parameters.

MSC: 46E35; 47B10

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1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $s_1, s_2 \in \mathbb{R}$ and $1 < p_1, p_2, q_1, q_2 < \infty$. It follows from a result of Pietsch [15] (see also [18] and [21, p. 354]) that the embedding $B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$ is nuclear if $s_1 - s_2 > d - d(1/p_2 - 1/p_1)_+$, and it is not nuclear if $s_1 - s_2 < d - d(1/p_2 - 1/p_1)_+$. Here $a_+ = \max\{a, 0\}$. The limiting case $s_1 - s_2 = d - d(1/p_2 -$

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$1/p_1)_+$ has been considered very recently by Edmunds, Gurka and Lang [7] and Triebel [27]. As one can see in [27], the embedding is not nuclear if $s_1 - s_2 = d - d(1/p_2 - 1/p_1)_+$. Note that the conditions above are independent of the parameters q_1 and q_2.

The aim of this paper is to investigate the role of q_1, q_2 in nuclearity of embeddings between Besov spaces. Having in mind the results on entropy numbers of limiting embeddings of Leopold [13] and Cobos and Kühn [4], we work with Besov spaces $B^{s,\alpha}_{p,q}(\Omega)$ which add logarithmic smoothness with exponent α to the classical smoothness s. It holds $B^{s,\alpha}_{p,q}(\Omega) \hookrightarrow B^{s}_{p,q}(\Omega)$ for $\alpha > 0$. We show a sufficient condition in terms of q_1, q_2 for nuclearity of embeddings $B^{s_1,\alpha_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2,\alpha_2}_{p_2,q_2}(\Omega)$ when $s_1 - s_2 = d - d(1/p_2 - 1/p_1)_+$. Furthermore, we prove that the condition is necessary for nuclearity in a wide range of parameters.

Our results not only point out the role of parameters q_1, q_2 in nuclearity of embeddings, but also apply to many embeddings $B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$ when $s_1 - s_2 = d - d(1/p_2 - 1/p_1)_+$ and $\min\{p_1, p_2, q_1, q_2\} = 1$ or/and $\max\{p_1, p_2, q_1, q_2\} = \infty$. These limiting embeddings have not been considered in [7] or [27].

The methods that we use are different from those by Edmunds, Gurka and Lang [7] and by Triebel [27]. The techniques of [7] use the relationship between nuclearity and s-numbers (Bernstein and approximation numbers), while those of [27] are based on wavelet expansions for functions of $B^{s}_{p,q}(\Omega)$, periodic function spaces on $\mathbb{T}^d$ and duality arguments. However, our methods only depend on the description in terms of wavelets for spaces $B^{s,\alpha}_{p,q}(\Omega)$ (see [1,4]) and the characterizations of nuclearity of diagonal operators between sequence spaces obtained by Tong [20].

2. Preliminaries

For $a \in \mathbb{R}$, we write $a_+ = \max\{a, 0\}$. Given two sequences $(a_j), (b_j)$ of non-negative real numbers we put $a_j \lesssim b_j$ if there is a constant $c > 0$ such that $a_j \le c\, b_j$ for all $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $a_j \lesssim b_j$ and $b_j \lesssim a_j$, we write $a_j \sim b_j$.

Let A_0, A_1 be Banach spaces continuously embedded in a Hausdorff topological vector space $\mathcal{A}$. Given $0 < \theta < 1$, $\gamma \in \mathbb{R}$ and $1 \le q \le \infty$, *the logarithmic interpolation space* $(A_0, A_1)_{\theta,\gamma,q}$ is formed by all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{(A_0,A_1)_{\theta,\gamma,q}} = \left(\int_0^\infty \big(t^{-\theta}(1 + |\log t|)^{-\gamma} K(t, a)\big)^q \frac{dt}{t} \right)^{1/q}$$

(as usual, the integral should be replaced by the supremum when $q = \infty$), where $K(t, a)$ is the K-functional of Peetre, defined by

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}$$

(see [10,8,5]). For $\gamma = 0$ we recover the classical real interpolation space $(A_0, A_1)_{\theta,q}$ realized as a K-space (see [2,21]). It is not hard to check that the logarithmic interpolation spaces have the interpolation property for bounded linear operators.

Let $1 \le p \le \infty$ and $M \in \mathbb{N}$. We let ℓ_p be the usual space of scalar p-summable sequences with indices in $\mathbb{N}_0$, and we write ℓ_p^M for the M-dimensional ℓ_p-space. For $1 \le q \le \infty$, $w_j > 0$ and $M_j \in \mathbb{N}$, the space $\ell_q(w_j \ell_p^{M_j})$ consists of all vector-valued sequences $x = (x_j)_{j \in \mathbb{N}_0}$ having a finite norm

$$\|x\|_{\ell_q(w_j \ell_p^{M_j})} = \left(\sum_{j=0}^{\infty} (w_j \|x_j\|_{\ell_p^{M_j}})^q \right)^{1/q}.$$