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Full Length Article

Christoffel transformations for multivariate orthogonal polynomials

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Abstract

Polynomial perturbations of real multivariate measures are discussed and corresponding Christoffel type formulas are found. The 1D Christoffel formula is extended to the multidimensional realm: multivariate orthogonal polynomials are expressed in terms of last quasi-determinants and sample matrices. The coefficients of these matrices are the original orthogonal polynomials evaluated at a set of nodes, which is supposed to be poised. A discussion for the existence of poised sets is given in terms of algebraic hypersurfaces in the complex affine space. Two examples of irreducible perturbations of total degree 1 and 2, for the bivariate product Legendre orthogonal polynomials, are discussed in detail.

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1. Introduction

In [1] we studied how the Gauss–Borel or LU factorization of a moment matrix allows for a better understanding of the links between multivariate orthogonal polynomials (MVOPR)

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on a multidimensional real space $\mathbb{R}^D$, $D \geq 1$, and integrable systems of Toda and KP type. In particular, it was shown how the LU decomposition allows for a simple construction of the three term relation or the Christoffel–Darboux formula. Remarkably, it is also useful for the construction of Miwa type expressions in terms of quasi-tau matrices of the MVOPR or the finding of the Darboux transformation of Christoffel type (Christoffel transformations in this paper). Indeed, we presented for the first time Christoffel transformations for orthogonal polynomials in several variables, that we called elementary, and its iteration, resulting in a Christoffel quasi-determinantal type formula. These Christoffel transformations allow for the construction of new MVOPR, associated with a perturbed measure, from the MVOPR of a given non perturbed measure. Observe that they also provide a direct method to construct new solutions of the associated Toda–KP type integrable systems.

What we called elementary Darboux transformations in [1] were given as the multiplication of the non perturbed measure by a degree one multivariate polynomial. The mth iteration of these so called elementary Darboux transformations leads, therefore, to a perturbation by a multivariate polynomial of degree m. This way of proceeding was motivated by the one dimensional situation, in that case happens that the irreducible polynomials have degree one (the fundamental theorem of algebra). But, in higher dimension the situation is much richer and we do have many irreducible polynomials of higher degree. Therefore, the territory explored for the Christoffel transformations in [1] was only a part, significant but incomplete, of a further more vast territory.

In this paper we give a generalization of the Christoffel formulas for the Christoffel transformations we discussed in [1], that holds for a perturbation by a polynomial of any degree, being prime or not prime. This provides us with an elegant quasi-determinantal expression for the new MVOPR which is a broad extension of the 1D determinantal Christoffel formula. For a detailed account of multivariate orthogonal polynomials see [4].

For the construction of the mentioned general Christoffel transformation (for the 1D scenario see [3,2,5]) we use multivariate interpolation theory, see [14]. Therefore, we need poised sets for which the sample matrix is not singular. In this paper we initiate the study of poised sets for general Darboux transformations. We find that the analysis can be split into two parts, one measure-independent part depending exclusively on the relative positions of nodes in the algebraic hypersurface of the generating polynomial, and another related to the non perturbed measure and the corresponding Jacobi matrices. The geometrical part, as usual in interpolation theory, requires of the concourse of Vandermonde matrices. In fact, of multivariate Vandermonde matrices, see [14], or multivariate confluent Vandermonde matrices.

With the aid of basic facts in algebraic geometry, see for example [10] or [18] we are able to show, for perturbing polynomials that can be expressed as the product $Q = Q_1 \cdots Q_N$ of N prime factors – see Theorem 3.3 – that there exists, in the complex domain, poised sets of nodes by forbidding its belonging to any further algebraic hypersurface, different from the algebraic hypersurface of Q, of lower degree. In the bivariate scenario, we analyze two specific prime perturbations of Legendre product polynomials. Moreover, we see that for a perturbation of the measure by a polynomial of the form $Q = \mathcal{R}^d$, poised sets never exist, and the Christoffel transformation as presented in Theorem 2.1 is not applicable. However, with the use of wronskian type matrices we can avoid this problem and find an appropriate extension of the Christoffel formula, see Theorem B.3, for a perturbing polynomial of the form $Q = Q_1^{d_1} \cdots Q_N^{d_N}$ where the polynomials Q_i are irreducible. The discussion on poised sets in this general scenario is given in Theorem B.4, where again the set of nodes when poised cannot belong to any further algebraic hypersurfaces of certain type.

The layout of the paper is as follows. We reproduce, for the reader commodity, some necessary material from [1]. In Section 2 we give the Christoffel transformation generated by a multivariate