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Journal of Approximation Theory 213 (2017) 70–77

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

A sharp bound on the Lebesgue constant for Leja points in the unit disk

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Received 19 July 2016; received in revised form 1 October 2016; accepted 21 October 2016
Available online 31 October 2016

Communicated by Vilmos Totik

Abstract

We give a sharp bound for the Lebesgue constant associated with Leja sequences in the complex unit disk, confirming a conjecture made by Calvi and Phung in 2011.

Keywords: Lagrange interpolation; Polynomial approximation; Lebesgue constant; Leja sequences

1. Introduction

Let K be a compact set in the complex plane. Leja sequences are defined by arbitrarily fixing a first point $e_0 \in K$ then by recursively selecting e_k, $k = 1, 2, \ldots$ such that

$$\prod_{j=0}^{k-1} |e_k - e_j| = \max_{z \in K} \prod_{j=0}^{k-1} |z - e_j|. \tag{1}$$

They were first studied by Edrei [6], then by Leja [8] who showed that the sequence $\left(\prod_{j=0}^{k-1} |e_k - e_j|\right)^{1/k}$ converges to the transfinite diameter of K.

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http://dx.doi.org/10.1016/j.jat.2016.10.004

Consider $\mathcal{C}(K)$ the space of complex valued continuous functions on K, endowed with the uniform norm. For any $f \in \mathcal{C}(K)$, the unique polynomial in the space Π_{k-1} of polynomials of degree at most $k-1$ which coincides with f on the underlying set of $E_k = (e_0, e_1, \dots, e_{k-1})$ is the Lagrange interpolation polynomial defined by

$$L_{E_k}(f)(z) = \sum_{j=0}^{k-1} f(e_j) l_{j,E_k}(z)$$

where

$$l_{j,E_k}(z) = \prod_{\substack{i=0 \\ i\neq j}}^{k-1} \frac{z-e_i}{e_j-e_i}, \quad j = 0, \dots, k-1$$

are the fundamental Lagrange interpolation polynomials.

The norm of L_{E_k}, as a continuous linear operator from $\mathcal{C}(K)$ onto Π_{k-1}, is the so-called Lebesgue constant

$$\Lambda_{E_k} := \sup_{\|f\|\le 1} \|L_{E_k}(f)\| = \sup_{z\in K} \lambda_{E_k}(z), \tag{2}$$

where

$$\lambda_{E_k}(z) := \sum_{j=0}^{k-1} |l_{j,E_k}(z)|.$$

The Lebesgue constant plays a crucial role in polynomial interpolation. The inequality

$$\|L_{E_k}(f) - f\|_K \le (\Lambda_{E_k} + 1) \inf_{P\in\Pi_{k-1}} \|f - P\|,$$

shows that it measures how close the interpolant is to the best polynomial approximant of a function. It also measures the stability of Lagrange interpolation. We refer the reader to [9] for more details and many interesting properties on the Lebesgue constant.

In the present paper, we are interested in finding an optimal bound for the Lebesgue constant associated with Leja points in the case where K is the complex unit disk: $\mathcal{U} = \{z \in \mathbb{C} : |z| \le 1\}$.

We consider Leja sequences $(e_k)_{k\ge 0}$ initiated at $e_0 \in \partial\mathcal{U} = \{|z| = 1\}$. There is no loss of generality in assuming that $e_0 = 1$ since any Leja sequence is the product by e_0 of a Leja sequence initiated at 1. We speak of Leja sequences rather than of a Leja sequence because at each step there may be several points e_k satisfying (1). Note that by the maximum principle, $|e_k| = 1,\ \ k = 1, 2, \dots$.

Any finite sequence $E_k = (e_0, \dots, e_{k-1})$ where $e_0, \dots, e_{k-1}$ are defined by (1) is called a k-Leja section.

Leja sequences of the disk were explicitly described by Białas-Cież and Calvi [1]. They showed that for any Leja sequence of the disk, initiated at $e_0 = 1$, the underlying set of the 2^nth section consists of the 2^nth roots of the unity. Besides, the 2^{n+1}th section is

$$(E^{2n}, \rho F^{2n}),$$

where ρ is any 2^nth root of -1 and $F_{2^n} = (f_0, \dots, f_{2^n-1})$ is a 2^n-Leja section with $f_0 = 1$, with the notation

$$(E_k, F_j) := (e_0, \dots, e_{k-1}, f_0, \dots, f_{j-1}).$$