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Breaking the curse for uniform approximation in Hilbert spaces via Monte Carlo methods

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ABSTRACT

We study the L_∞-approximation of d-variate functions from Hilbert spaces via linear functionals as information. It is a common phenomenon in tractability studies that unweighted problems (with each dimension being equally important) suffer from the curse of dimensionality in the deterministic setting, that is, the number $n(\epsilon, d)$ of information needed in order to solve a problem within a given accuracy $\epsilon > 0$ grows exponentially in d. We show that for certain approximation problems in periodic tensor product spaces, in particular Korobov spaces with smoothness $r > 1/2$, switching to the randomized setting can break the curse of dimensionality, now having polynomial tractability, namely $n(\epsilon, d) \preceq \epsilon^{-2} d\,(1 + \log d)$. Similar benefits from Monte Carlo methods in terms of tractability have only been known for integration problems so far.

1. Introduction

Concerning the problem of computing the integral $\mathrm{INT}(f) := \int_{[0,1]^d} f(\mathbf{x})\,d\mathbf{x}$ of a function $f : [0,1]^d \to \mathbb{R}$ based on information from n function evaluations, it is known that for many classes of input functions deterministic methods suffer from the *curse of dimensionality*, that is, the number $n^{\mathrm{det}}(\varepsilon, d)$ of function values needed in order to guarantee some given accuracy $\varepsilon > 0$ grows exponentially in d. One example of such input classes where the curse holds, are C^r-functions with bounded partial derivatives, see Hinrichs et al. [12]. Another type of problems can be formulated with functions that are bounded in the norm of certain unweighted tensor product Hilbert spaces,

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see Novak and Woźniakowski [25, Chapter 16]. Usually, the standard Monte Carlo method,

$$M_n(f) := \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{X}_i), \qquad \mathbf{X}_i \overset{\text{iid}}{\sim} \text{unif}([0,1]^d), \tag{1}$$

can be used to bound the complexity in the randomized setting, $n^{\text{ran}}(\varepsilon, d) \le \lceil \varepsilon^{-2} \rceil$, if functions under consideration are bounded in the L_2-norm, $\|f\|_2 \le 1$. Offering an upper bound which is independent from d and polynomial in ε^{-1}, these problems are *strongly polynomially tractable* when using Monte Carlo. It has been unknown so far whether there exist d-variate function approximation problems where Monte Carlo methods may help in a similar way. Function approximation problems seem to be more difficult than integration problems, because instead of just a single real number (namely the integral value) we aim to find a representation for a function. There are examples of function approximation problems where Monte Carlo methods do not improve tractability, namely uniform approximation of certain classes of C^∞-functions, see [16], or [17, Section 2.4.2] for an extended discussion. In contrast, the aim of the present paper is to present an example of function approximation problems where the curse of dimensionality does hold for deterministic methods, but Monte Carlo methods achieve *polynomial tractability* with $n^{\text{ran}}(\varepsilon, d) \le \varepsilon^{-2}\, d\,(1 + \log d)$, hence *breaking the curse*. Somewhere in between integration and approximation is the paper of Heinrich and Milla [11] on indefinite integration of L_p-functions, where the output is Banach space valued. In that setting only randomized methods can solve the problem — and actually provide polynomial tractability.

In this paper we study the uniform approximation in Hilbert spaces $\mathcal{H}^d$ of functions on a d-dimensional domain, $\text{APP} : \mathcal{H}^d \hookrightarrow L_\infty$, based on information from n linear functionals. This type of embeddings is a special case of a linear problem $S : F \to G$ where F and G are Banach spaces. (For simplicity, we restrict to Banach spaces over the reals.) A general deterministic approximation method for such a problem is a map $A_n : F \overset{N}{\to} \mathbb{R}^n \overset{\phi}{\to} G$ with an information operator $N(f) = (L_1(f), \ldots, L_n(f))$, where the L_i are continuous linear functionals acting on F. (One could think about choosing the information functionals adaptively, but this will not be necessary in the context of this paper.) The error of a deterministic method is defined by the worst case,

$$e(A_n, S) := \sup_{\|f\|_F \le 1} \|Sf - A_n(f)\|_G.$$

Randomized methods are families $(A_n^\omega)_\omega$ of such mappings, indexed by a random element $\omega \in \Omega$ from a suitable probability space $(\Omega, \Sigma, \mathbb{P})$, we suppose sufficient measurability. The error for a single input is now averaged over ω, we define

$$e((A_n^\omega)_\omega, S) := \sup_{\|f\|_F \le 1} \mathbb{E}\|Sf - A_n^\omega(f)\|_G.$$

For the two algorithmic settings, the *nth minimal error*, that is the error achievable by optimal algorithms that use n pieces of information, is denoted by

$$e^{\text{det}}(n, S) := \inf_{A_n} e(A_n, S), \qquad \text{and} \qquad e^{\text{ran}}(n, S) := \inf_{(A_n^\omega)_\omega} e((A_n^\omega)_\omega, S).$$

The *initial error* $e(0, S) = \|S\|_{F\to G}$ coincides for both settings. Tractability studies put the focus on the inverse notion *ε-complexity*, namely, the *(worst case) deterministic complexity* $n^{\text{det}}(\varepsilon, S)$ is the minimal $n \in \mathbb{N}_0$ for which we can find a deterministic algorithm A_n that guarantees an error smaller than or equal $\varepsilon > 0$, analogously we have the *Monte Carlo complexity* $n^{\text{ran}}(\varepsilon, S)$ using randomized algorithms. In tractability, the complexity (in either algorithmic setting) is regarded as a function $n(\varepsilon, d) = n(\varepsilon, S^d)$ for families $(S^d)_{d\in\mathbb{N}}$ of problems with dimensional parameter d. For more details on these and related notions from *information-based complexity* (IBC), see the books [24,32].

Section 2 is a collection of abstract tools for understanding L_∞-approximation in Hilbert spaces of functions on general domains. Precisely, these spaces have to be *reproducing kernel Hilbert spaces* (RKHS), see Section 2.2 for a brief overview. The upper Monte Carlo bound we use links the Monte Carlo complexity to the expected maximum norm of the Gaussian process Ψ associated with the given RKHS,

$$n^{\text{ran}}(n, \mathcal{H} \hookrightarrow L_\infty) \le \lceil 4\,(\mathbb{E}\|\Psi\|_\infty)^2\, \varepsilon^{-2} \rceil,$$

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