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The rank of random regular digraphs of constant degree

Alexander E. Litvak [a,*], Anna Lytova [b], Konstantin Tikhomirov [c], Nicole Tomczak-Jaegermann [a], Pierre Youssef [d]

[a] Department of Math. and Stat. Sciences, University of Alberta, Edmonton, AB, Canada, T6G 2G1
[b] Faculty of Math., Physics, and Comp. Science, University of Opole, ul. Oleska 48, 45-052, Opole, Poland
[c] Department of Math., Princeton University, Fine Hall, Washington road, Princeton, NJ 08544, United States
[d] Université Paris Diderot, Laboratoire de Probabilités et de modèles aléatoires, 75013 Paris, France

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ABSTRACT

Let d be a (large) integer. Given $n \geq 2d$, let A_n be the adjacency matrix of a random directed d-regular graph on n vertices, with the uniform distribution. We show that the rank of A_n is at least $n-1$ with probability going to one as n grows to infinity. The proof combines the well known method of simple switchings and a recent result of the authors on delocalization of eigenvectors of A_n.

1. Introduction

Singularity of random discrete square matrices is a subject with a long history and many results and applications. In particular, quantitative estimates on the smallest singular number are important for understanding complexity of some algorithms. Well invertible sparse matrices are of general interest in computer science, and it is known that sparse matrices are computationally more efficient (require less operations for matrix–vector multiplication). In this paper we deal with sparse random square matrices from a certain model.

In a standard setting, when the entries of the $n \times n$ matrix are i.i.d. Bernoulli ± 1 random variables, the invertibility problem has been addressed by Komlós in [11,12], and later considered in several papers [10,21,4]. A long-standing conjecture asserts that the probability that the Bernoulli matrix

* Corresponding author.
E-mail addresses: aelitvak@gmail.com (A.E. Litvak), alytova@math.uni.opole.pl (A. Lytova), kt12@math.princeton.edu (K. Tikhomirov), nicole.tomczak@ualberta.ca (N. Tomczak-Jaegermann), youssef@math.univ-paris-diderot.fr (P. Youssef).

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is singular is $\left(1/2+o(1)\right)^n$. Currently, the best upper bound on this probability is $\left(1/\sqrt{2}+o(1)\right)^n$ obtained by Bourgain, Vu, and Wood [4]. We would also like to mention related works on singularity of symmetric Bernoulli matrices [7,19,22] and Nguyen's work [20], where random 0/1 matrices with independent rows and row-sums constraints were considered.

A corresponding question can be formulated for adjacency matrices of random graphs. For instance, consider the adjacency matrix of an undirected Erdős–Renyi random graph $G(n,p)$ which is a symmetric random $n\times n$ matrix whose off-diagonal entries are i.i.d. 0/1 random variables with the parameter p. The case $p=1/2$ is closely related to the random model from the previous paragraph. In [8] Costello and Vu proved that, given $c>1$, with large probability the rank of the adjacency matrix of $G(n,p)$ is equal to the number of non-isolated vertices whenever $c\ln n/n\le p\le 1/2$. It is known that $p=\ln n/n$ is the threshold of connectivity, so that when $c>1$ and $c\ln n/n\le p\le 1/2$, the graph $G(n,p)$ typically contains no isolated vertices and is therefore of full rank with probability going to one as n tends to infinity (see [1] for quantitative bounds in the non-symmetric setting). It was also shown that if $p\to 0$ and $np\to\infty$, then $\left(\operatorname{rk} G(n,p)\right)/n\to 1$ as n goes to infinity, where $\operatorname{rk}(A)$ stands for the rank of the matrix A. The case $p=y/n$ for a fixed y was studied in [3] where asymptotics for $\left(\operatorname{rk} G(n,p)\right)/n$ were established.

In the absence of independence between the matrix entries, the problem of singularity involves additional difficulties. Such a problem was considered for the (symmetric) adjacency matrix M_n of a random (with respect to the uniform probability) undirected d-regular graph on n vertices, i.e., a graph in which each vertex has precisely d neighbours. The case $d=1$ corresponds to a permutation matrix which is non-singular, and for $d=2$ the graph is a union of cycles and the matrix is almost surely singular. Moreover, the invertibility of the adjacency matrix of the complementary graph is equivalent to that of the original one (in fact, the ranks of the adjacency matrices of a d-regular graph and of its complementary graph are the same). This can be seen by first noticing that the eigenvalues of J_n-M_n, where J_n is the $n\times n$ matrix of ones, are equal to the difference between those of J_n and those of M_n (since the two commute) and that all eigenvalues of M_n are bounded in absolute value by d, which is smaller than the only non-zero eigenvalue of J_n (equals to n). In parallel to the Erdős–Renyi model, Costello and Vu raised the following problem: "For what d is the adjacency matrix M_n of full rank almost surely?" (see [8, Section 10]). They conjectured that for every $3\le d\le n-3$, the adjacency matrix M_n is non-singular with probability going to 1 as n tends to ∞. This conjecture was mentioned again in the survey [23, Problem 8.4] and 2014 ICM talks by Frieze [9, Problem 7] and by Vu [24, Conjecture 5.8].

In the present paper, we are interested in behaviour of adjacency matrices of random directed d-regular graphs with the uniform model, that is, random graphs uniformly distributed on the set of all directed d-regular graphs on n vertices. By a directed d-regular graph on n vertices we mean a graph such that each vertex has precisely d in-neighbours and d out-neighbours and where loops and 2-cycles are allowed but multiple edges are prohibited. The adjacency matrix A_n of such a graph is uniformly distributed on the set of all (not necessarily symmetric) 0/1 matrices with d ones in every row and every column. As in the symmetric case, in the case $d=1$ the matrix A_1 is a permutation matrix which is non-singular, and in the case $d=2$ the matrix A_2 is almost surely singular. It is natural to ask the same question as in [8] for directed d-regular graphs (see, in particular, [5, Conjecture 1.5]). Cook [5] proved that such a matrix is asymptotically almost surely non-singular for $\omega(\ln^2 n)\le d\le n-\omega(\ln^2 n)$, where $f=f(n)=\omega(a_n)$ means $f/a_n\to\infty$ as $n\to\infty$. Further, in [13,14], the authors of the present paper showed that the singularity probability is bounded above by $C\ln^3 d/\sqrt{d}$ for $C\le d\le n/\ln^2 n$, where C is a (large) absolute positive constant. This settles the problem of singularity for $d=d(n)$ growing to infinity with n at any rate. Moreover, quantitative bounds on the smallest singular value for this model were derived in [6] and [17]. Those estimates turn out to be essential in the study of the limiting spectral distribution [6,15].

The challenging case when d is a constant remains unresolved and is the main motivation for writing this note. The lack of results in this setting constitutes a major obstacle in establishing the conjectured non-symmetric (oriented) Kesten–McKay law as the limit of the spectral distribution for the directed random d-regular graph (see, in particular, [2, Section 7]). This note illustrates a partial progress in this direction. Our main result is the following theorem. Note that the probability bound in it is non-trivial only if $\ln n>C\ln^2 d$, however in the complementary case we have $\operatorname{rk}(A_n)=n$ with high probability as was mentioned above.

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