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An upper bound on the smallest singular value of a square random matrix ☆

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ABSTRACT

Let $A = (a_{ij})$ be a square $n \times n$ matrix with i.i.d. zero mean and unit variance entries. It was shown by Rudelson and Vershynin in 2008 that the upper bound for the smallest singular value $s_n(A)$ is of order $n^{-\frac{1}{2}}$ with probability close to one under the additional assumption that the entries of A satisfy $\mathbb{E}a_{11}^4 < \infty$. We remove the assumption on the fourth moment and show the upper bound assuming only $\mathbb{E}a_{11}^2 = 1$.

1. Introduction

The extremal singular values have been attracting the attention of scientists in different disciplines such as mathematical physics or geometric functional analysis. In particular, they play an important role in numerical analysis as the condition number, which is the ratio of the largest to the smallest singular value, is a measure for the worst-case loss of precision in a computational problem. Much is known about the behavior of the largest singular value and we refer the reader to [2,25,39]. The study of the behavior of the smallest singular value goes back to von Neumann and his collaborators concerning numerical inversion of large matrices, where they conjectured (see [34,35]) that the smallest singular value is of order $n^{-\frac{1}{2}}$ with probability close to one. Estimates of similar type for the case of Gaussian matrices (i.e., matrices with i.i.d. standard normal entries) were obtained by Edelman in [6] and Szarek in [27]. For estimates on extremal singular values which were acquired while studying the problem of the approximation of covariance matrices, we refer to [1,8,15,33]. Various bounds for the smallest singular value have been obtained under rather weak assumptions

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on the rows of the matrix in [9,18,37,38]. For lower bounds on the smallest singular value of random matrices with independent but not identically distributed entries see a recent result by Cook [4]. We refer readers with further interest in this topic to [3,5,16,28,30,31,32].

Rudelson and Vershynin in [23,22,24] studied the behavior of the smallest singular value of matrices with i.i.d. subgaussian entries. They showed (see [23,22]) that the smallest singular value of a square random matrix A with i.i.d. subgaussian entries is of order $n^{-\frac{1}{2}}$. In particular, in [23] they proved that for given $t \geq 2$ there are $C > 0$ and $u \in (0, 1)$ depending only on the subgaussian moment of entries of A such that

$$\mathbb{P}\left(s_n(A) > tn^{-\frac{1}{2}}\right) \leq C\frac{\log t}{t} + u^n.$$

Nguyen and Vu in [17] showed an exponential bound for the above probability, which improves the linear bound by Rudelson and Vershynin. A lower bound for rectangular subgaussian matrices was obtained in [24]. A recent result of Wei (see [36]) provides upper bounds on intermediate singular values of rectangular matrices with subgaussian entries. The corresponding lower bounds were obtained in [20].

Recently, in [19] a new technique was developed, which allowed Rebrova and Tikhomirov to prove a lower bound for $s_n(A)$ of square matrices of order $n^{-\frac{1}{2}}$ under the assumption that the Lévy concentration function of entries of A is bounded. Namely, they showed the small ball probability estimate:

$$\forall \varepsilon > 0 : \quad \mathbb{P}\left(s_n(A) \leq \varepsilon n^{-\frac{1}{2}}\right) \leq C\varepsilon + u^n,$$

where $C > 0$ and $u \in (0, 1)$ depend only on the law of a_{11}. Notice that any random variable ξ with $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$ has a bounded Lévy concentration function, therefore the above statement is valid for matrices with assumptions only on the second moment of entries.

The goal of this note is to show that the upper bound on the smallest singular value holds for square matrices with heavy-tailed entries. We prove the following theorem.

Theorem 1.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries are i.i.d. random variables with $\mathbb{E}a_{ij} = 0$ and $\mathbb{E}a_{ij}^2 = 1$. Then there exists an absolute constant $C > 0$ such that for every $\varepsilon > 0$*

$$\mathbb{P}\left(s_n(A) > \frac{1}{\varepsilon^2}n^{-\frac{1}{2}}\right) \leq C\varepsilon + \frac{C}{\sqrt{n}}.$$

We expect that the dependence on ε can be improved to ε^{-1}, but our proof gives only ε^{-2}.

We now briefly describe the ideas of proof of Theorem 1.1.

To estimate the smallest singular value of a random matrix A we will use the following equivalence, which holds for every $\lambda \geq 0$,

$$s_n(A) \leq \lambda \quad \Longleftrightarrow \quad \exists x \in S^{n-1} : \ \|Ax\| \leq \lambda.$$

We will show that there exists $x \in \mathbb{R}^n$ such that $\|x\| \leq \tau$ and $\|A^{-1}x\| \geq \eta\sqrt{n}$ for some $\tau, \eta > 0$, which implies $s_n(A) \leq \frac{\tau}{\eta\sqrt{n}}$. Let us describe the main difficulty in our proof. It is well-known that $A^{-1}x$ behaves differently depending on the structure of x. We follow [12,13] and roughly speaking split the unit sphere into two parts consisting of vectors of small dimensions and vectors with bounded ℓ_∞ norm. To deal with vectors of the second type, we use ideas introduced in [22], namely we use the essential least common denominator (see the definition below). Denote by B the transpose of the first $n-2$ columns of matrix A. To show that the essential least common denominator of vectors in the null space of a matrix B has exponential decay with high probability, in [23] the authors used a standard ε-net argument, namely, for a given ε-net $\mathcal{N}$ on a subset $S \subset S^{n-1}$ one has

$$\inf_{y \in S} \|By\| \geq \inf_{y' \in \mathcal{N}} \left(\|By'\| - \|B\|\|y - y'\|\right).$$

This procedure relies on an upper bound for the operator norm $\|B\|$, which is of order $n^{\frac{1}{2}}$ with exponentially high probability under the subgaussian moment assumption on the entries of B.

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