Journal of Complexity 47 (2018) 62–85

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Novel results for the anisotropic sparse grid quadrature[☆]

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ARTICLE INFO

Article history:
Received 20 February 2017
Accepted 7 February 2018
Available online 21 February 2018

Keywords:
Multivariate quadrature
Smolyak's construction
Sparse grids
Complexity
Curse of dimension

ABSTRACT

This article is dedicated to the anisotropic sparse grid quadrature for functions which are analytically extendable into an anisotropic tensor product domain. Taking into account this anisotropy, we end up with a dimension independent error versus cost estimate of the proposed quadrature. In addition, we provide a novel and improved estimate for the cardinality of the underlying anisotropic index set. To validate the theoretical findings, we present several examples ranging from simple quadrature problems to diffusion problems on random domains. These examples demonstrate the remarkable convergence behavior of the anisotropic sparse grid quadrature in applications.

1. Introduction

This article is dedicated to the construction of anisotropic sparse grid quadrature methods for functions which are analytically extendable into an anisotropic tensor product domain. Specifically, we will develop and analyze a particular realization based on Gauss–Legendre quadrature rules. Anisotropic sparse grid quadrature methods can be seen as a generalization of sparse Smolyak type quadratures, cf. [31], since they are explicitly tailored to the anisotropic behavior of the underlying

☆ Communicated by Klaus Ritter.

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https://doi.org/10.1016/j.jco.2018.02.003
0885-064X/

integrand. Taking into account these anisotropies leads to a remarkable improvement in the cost of the sparse grid quadrature.

Usually, a sparse grid quadrature is described by some sparse index set and a sequence of univariate quadrature rules. For the sequence of univariate quadrature rules, we employ Gauss–Legendre quadratures with linearly increasing numbers of quadrature points. The index set is priorly chosen with respect to a certain weight vector, which incorporates the anisotropy, and a predefined approximation level. It is also possible to adaptively select those indices which provide the main contribution to the integral, see [11]. Such adaptive methods have successfully been applied in the context of random diffusion problems, see e.g. [7,23,28], in order to compute best N-term approximations of the corresponding solution. However, the adaptive construction of index sets is computational expensive and only heuristic error indicators are available. Hence, it can not be guaranteed that the adaptively selected index set is optimal. Instead of choosing Gauss–Legendre points, a sequence of nested quadrature rules such as Clenshaw–Curtis or Leja type quadratures could be considered as well. While the number of quadrature points needs to be doubled for Clenshaw–Curtis quadratures in order to guarantee their nestedness, only one additional quadrature point is added for each consecutive member of the Leja sequence. Hence, based on the Leja sequence, a sparse grid quadrature can be constructed where only one additional function evaluation is required for each new multi-index in the sparse index set, cf. [13]. However, in contrast to Gaussian quadratures, the quadrature weights of the Leja sequence are not necessarily positive which yields that the stability constant of the quadrature might not be uniformly bounded. Moreover, Gaussian quadrature rules provide a much higher degree of polynomial exactness than Leja quadratures with the same number of quadrature points, which is particularly advantageous for smooth integrands.

The main task in estimating the quadrature's cost is the estimation of the number of multi-indices which are contained in the sparse index set. For the isotropic variant, the number of indices can easily be determined by combinatorial arguments, see e.g. [10,26,32]. Things get more involved if one considers anisotropic, i.e. weighted, sparse index sets, which yields a particular instance of a weighted tensor product algorithm, see [33] and especially [27, Chapter 15], where a comprehensive overview of related literature can be found. In this case, to the best of our knowledge, only very rough estimates on the cardinality of the index set are known, although several estimates can be found in the literature, see e.g. [4]. In fact, this problem is equivalent to the estimation of the number of integer solutions to linear Diophantine inequalities, see [29] and the references therein, which is a problem in number theory, or to the calculation of the integer points in a convex polyhedron. Current estimates are not sharp and do not provide improved results for the cost of the anisotropic sparse grid quadrature in comparison with the anisotropic full tensor product quadrature. In this article, we prove a novel formula to estimate the cardinality of the sparse index set in the weighted case. This formula is much more accurate than the other established estimates.

A very popular application that requires efficient high-dimensional quadrature rules are parametric partial differential equations. They are obtained, for example, from partial differential equations with random data by truncating the series expansions of the underlying random fields and parametrizing them with respect to the random fields' distribution. As representatives for such problems, we shall consider here elliptic diffusion problems with random coefficients as well as diffusion problems on random domains as specific examples to quantify the performance of the anisotropic sparse grid quadrature. The resulting quadrature approach is very similar to the anisotropic sparse grid collocation method based on Gaussian collocation points which has been introduced in [24,25]. The collocation method interpolates the random solution in certain collocation points and represents it in the parameter space with the aid of polynomials. Thus, it belongs to the class of non-intrusive methods, cf. [1]. Instead of representing the random solution itself, the anisotropic sparse grid quadrature can be employed to directly compute the solutions statistics, i.e. its moments, and functionals of the solution.

The remainder of this article is organized as follows. Section 2 specifies the quadrature problem under consideration and provides the corresponding framework. The subsequent Section 3 is dedicated to the construction of the anisotropic sparse grid quadrature method. In particular, the main ingredients, i.e. the index set and the sequence of univariate quadrature rules are introduced. In Section 4, we provide corresponding error estimates with respect to the level of the anisotropic sparse