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Asymptotically tight worst case complexity bounds for initial-value problems with nonadaptive information☆,☆☆

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ABSTRACT

It is known that, for systems of initial-value problems, algorithms using adaptive information perform much better in the worst case setting than the algorithms using nonadaptive information. In the latter case, lower and upper complexity bounds significantly depend on the number of equations. However, in contrast with adaptive information, existing lower and upper complexity bounds for nonadaptive information are not asymptotically tight. In this paper, we close the gap in the complexity exponents, showing asymptotically matching bounds for nonadaptive standard information, as well as for a more general class of nonadaptive linear information.

1. Introduction

We aim at closing a gap between upper and lower worst case complexity bounds for initial-value problems with nonadaptive information. A motivation comes from a discussion on this subject that we had with Stefan Heinrich in 2016. We deal with the solution of systems

$$z'(t) = f(z(t)), \quad t \in [a, b], \qquad z(a) = \eta, \tag{1}$$

where $a < b, f : \mathbf{R}^d \to \mathbf{R}^d$ is a C^r function, $d \geq 1$, and $\eta \in \mathbf{R}^d$. A class of functions f, denoted by $F_{r,d}$, is given by (2). For $\varepsilon > 0$, we measure the difficulty of the problem by the minimal cost of an algorithm,

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based on some information, that gives us an ε-approximation to the solution (the ε-complexity of the problem). If adaptive information is allowed, then the ε-complexity is denoted by $\mathrm{comp}(\varepsilon, F_{r,d})$. The notation $\mathrm{comp}^{\mathrm{nonad}}(\varepsilon, F_{r,d})$, where the superscript is added, means that we restrict ourselves to the class of nonadaptive information. For precise definitions of basic notions, the reader is referred to the next section. Our aim is to establish the asymptotics of $\mathrm{comp}^{\mathrm{nonad}}(\varepsilon, F_{r,d})$ as $\varepsilon \to 0$ for nonadaptive information, as function of the regularity r and the dimension d.

A question about potential advantages of adaptive over nonadaptive algorithms for solving various problems is an important issue in numerical analysis. Many different points of view cause some discussions and sometimes misunderstandings among numerical analysts in that respect. From practical point of view, adaption is claimed to be definitely better, which is supported by results of numerical experiments, see e.g. [2,6,7] and many other papers. A closer look however shows that advantages of adaption depend very much on the problem itself and the class of problem instances being solved. It is not a purpose of this paper to discuss the adaption/nonadaption issue in details — to have a flavor of it, one can consult the monograph [9], or recent papers [1,5,8].

In what follows, for a positive function $\gamma = \gamma(\varepsilon)$, the asymptotic expressions $O(\gamma(\varepsilon))$, $\Omega(\gamma(\varepsilon))$ and $\Theta(\gamma(\varepsilon))$ will always be meant as $\varepsilon \to 0$. It is known for many years that for problem (1) adaptive information is much more efficient in the worst case setting than nonadaptive one. It was shown for adaptive information that the ε-complexity of (1) is, (see [3]):

$$\mathrm{comp}(\varepsilon, F_{r,d}) = \Theta\left((1/\varepsilon)^{1/r}\right), \qquad \text{for the class of standard adaptive information,}$$
$$\mathrm{comp}(\varepsilon, F_{r,d}) = \Theta\left((1/\varepsilon)^{1/(r+1)}\right), \qquad \text{for the class of linear adaptive information.}$$

In both cases of standard and linear information, the complexity bounds are asymptotically tight, and the asymptotics is independent of d.

In the nonadaptive case, the existing complexity bounds are not tight. In [4], we considered the class $F_{r,d}$ with $M = (0,1)^d$ and $D = 1$, see (2). It was shown (translating the results from non-autonomous problems in [4] to the autonomous ones (1)) that

$$(a)\quad \mathrm{comp}^{\mathrm{nonad}}(\varepsilon, F_{r,d}) = \Omega\left((1/\varepsilon)^{d/(r+1)}\right), \qquad \text{for the class of all linear nonadaptive information,}$$
$$(b)\quad \mathrm{comp}^{\mathrm{nonad}}(\varepsilon, F_{r,d}) = O\left((1/\varepsilon)^{d/r}\right), \qquad \text{for the class of standard nonadaptive information.}$$

The influence of the dimension d in the nonadaptive case is very significant, which indicates that the problem (1) is not well suited for nonadaptive solution. The complexity radically increases (asymptotically) with d.

The bounds *(a)* and *(b)* do not match, so that the asymptotics of the ε-complexity of the nonadaptive solution of (1) is not known. In this paper, we close the gap between lower and upper bounds in some important cases. We show that for $d \geq 2$

$$\mathrm{comp}^{\mathrm{nonad}}(\varepsilon, F_{r,d}) = \Omega\left((1/\varepsilon)^{(d-1)/r}\right), \qquad \text{for the class of all linear nonadaptive information.}$$

For $d > r+1$ this is an improvement over the lower bound *(a)*. It shows in particular that it is not possible in general, as one may expect, to achieve the complexity proportional to $(1/\varepsilon)^{d/(r+1)}$ by allowing nonadaptive linear (nonstandard) information. Our main result, contained in Theorem 1 and next extended in Theorem 2, states that

$$\mathrm{comp}^{\mathrm{nonad}}(\varepsilon, F_{r,d}) = \Omega\left((1/\varepsilon)^{d/r}\right), \qquad \text{for a class of linear nonadaptive information that includes any standard information.}$$

This improves the lower bound *(a)*, and matches the upper bound *(b)*. The question about the asymptotics of the ε-complexity for the class of all linear nonadaptive information is still open. It is thought to be $\Theta\left((1/\varepsilon)^{d/r}\right)$ as $\varepsilon \to 0$, the same as for standard information. Finally, in Remark 1 we point out how the proof of Theorem 1 can be modified to get the complexity lower bound for non-autonomous systems. In the lower bound of Theorem 1, one needs to replace d by $d+1$ in the exponent.

The paper is organized as follows. In Section 2 basic notation is established and known results are recalled. Section 3 is devoted to the proof of the main result in Theorem 1, and to its extension in Theorem 2. In the Appendix we give auxiliary constructions and bounds.