ARTICLE IN PRESS

Journal of Complexity ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Covering and separation of Chebyshev points for non-integrable Riesz potentials[☆]

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ARTICLE INFO

Article history:
Received 2 March 2017
Accepted 21 November 2017
Available online xxxx

Keywords:
Covering radius
Best covering
Separation distance
Maximal Riesz polarization
Chebyshev constant

ABSTRACT

For Riesz s-potentials $K(x, y) = |x - y|^{-s}$, $s > 0$, we investigate separation and covering properties of N-point configurations $\omega_N^* = \{x_1, \ldots, x_N\}$ on a d-dimensional compact set $A \subset \mathbb{R}^\ell$ for which the minimum of $\sum_{j=1}^N K(x, x_j)$ is maximal. Such configurations are called N-point optimal Riesz s-polarization (or Chebyshev) configurations. For a large class of d-dimensional sets A we show that for $s > d$ the configurations ω_N^* have the optimal order of covering. Furthermore, for these sets we investigate the asymptotics as $N \to \infty$ of the best covering constant. For these purposes we compare best-covering configurations with optimal Riesz s-polarization configurations and determine the sth root asymptotic behavior (as $s \to \infty$) of the maximal s-polarization constants. In addition, we introduce the notion of "weak separation" for point configurations and prove this property for optimal Riesz s-polarization configurations on A for $s > \dim(A)$, and for $d - 1 \leqslant s < d$ on the sphere $\mathbb{S}^d$.

1. Introduction

Suppose A is a compact subset of a Euclidean space $\mathbb{R}^\ell$ and $\omega_N = \{x_1, \ldots, x_N\} \subset A$ is a *multiset* (or an *N-point configuration*); i.e., a set of points with possible repetitions and cardinality $\#\omega_N = N$,

☆ Communicated by I. Sloan.

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https://doi.org/10.1016/j.jco.2017.11.007
0885-064X/© 2017 Published by Elsevier Inc.

counting multiplicities. For a positive number s we put

$$P_s(A;\omega_N) := \inf_{y\in A}\sum_{j=1}^{N}\frac{1}{|y-x_j|^s}.$$

Then the *Nth s-polarization (or Chebyshev) constant of A* is defined by

$$\mathcal{P}_s(A;N) := \sup_{\omega_N\subset A} P_s(A;\omega_N).$$

We note that since A is compact, there exists for each $N\in\mathbb{N}$ a configuration $\omega_N^* = \{x_1^*,\dots,x_N^*\}$ and a point y^* such that

$$\mathcal{P}_s(A;N) = P_s(A;\omega_N^*) = \sum_{j=1}^{N}\frac{1}{|y^*-x_j^*|^s}. \tag{1}$$

We call ω_N^* an *optimal (or extremal) Riesz s-polarization configuration* or simply an *optimal configuration*.

From an applications prospective, the maximal polarization problem, say on a compact surface (or body), can be viewed as the problem of determining the smallest number of sources (injectors) of a substance together with their optimal locations that can provide a required saturation of the substance at every point of the surface (body).

The general notion of polarization (or Chebyshev constants) for potentials was likely first introduced by Ohtsuka [18]. Further investigations of the asymptotic behavior as $N\to\infty$ of polarization constants as well as the asymptotic behavior of optimal configurations appear, for example, in [1,8,10,9,2,20,4,3,19].

The following result is a special case of a theorem due to Borodachov, Hardin, Reznikov and Saff [3] (see also [4]). It describes the asymptotic behavior of optimal configurations for the case of non-integrable Riesz kernels on A. Here and throughout we denote by $\mathcal{H}_d$ the Hausdorff measure on $\mathbb{R}^\ell$, $d\leqslant\ell$, normalized by $\mathcal{H}_d([0,1]^d)=1$, where $[0,1]^d$ is the d-dimensional unit cube embedded in $\mathbb{R}^\ell$.

Theorem 1.1. *Suppose A is a compact C^1-smooth d-dimensional manifold, embedded in $\mathbb{R}^\ell$ with $d\leqslant\ell$, and $\mathcal{H}_d(\partial A)=0$, where ∂A denotes the boundary of A. If $s>d$, then there exists a positive finite constant $\sigma_{s,d}$ that does not depend on A such that*

$$\lim_{N\to\infty}\frac{\mathcal{P}_s(A;N)}{N^{s/d}} = \frac{\sigma_{s,d}}{\mathcal{H}_d(A)^{s/d}}. \tag{2}$$

Moreover, if $\{\omega_N^*\}_{N=1}^\infty$ is any sequence of optimal configurations satisfying (1), then the normalized counting measures μ_N^* for the multisets ω_N^* satisfy

$$\mu_N^* := \frac{1}{N}\sum_{x\in\omega_N^*}\delta_x \xrightarrow{*} \mu,$$

where $\xrightarrow{*}$ denotes convergence in the weak topology, and μ is the uniform measure on A; i.e., for any Borel set $B\subset\mathbb{R}^\ell$*

$$\mu(B) = \frac{\mathcal{H}_d(B\cap A)}{\mathcal{H}_d(A)}.$$

In other words, in the limit, optimal polarization configurations ω_N^* for non-integrable Riesz potentials are uniformly distributed in the weak* sense. In this paper we study more distributional properties of optimal configurations ω_N^*. In particular, we investigate their separation, their covering (or mesh) radius, and their connection to the "best covering problem" for the set A.

Definition 1.2. Let A be a compact subset of a Euclidean space. For any N-point configuration $\omega_N\subset A$, the *separation constant* of ω_N is defined by

$$\delta(\omega_N) := \min_{i\neq j}|x_i-x_j|$$

Please cite this article in press as: A. Reznikov, et al., Covering and separation of Chebyshev points for non-integrable Riesz potentials, Journal of Complexity (2017), https://doi.org/10.1016/j.jco.2017.11.007.