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Monte Carlo methods for uniform approximation on periodic Sobolev spaces with mixed smoothness☆

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ABSTRACT

We consider the order of convergence for linear and nonlinear Monte Carlo approximation of compact embeddings from Sobolev spaces of dominating mixed smoothness with integrability $1 < p < \infty$ defined on the torus $\mathbb{T}^d$ into the space $L_\infty(\mathbb{T}^d)$ via methods that use arbitrary linear information. These cases are interesting because we can gain a speedup of up to 1/2 in the main rate compared to deterministic approximation. In doing so we determine the rate for some cases that have been left open by Fang and Duan (2007, 2008).

1. Introduction

Nowadays, Monte Carlo methods are widely used in many areas of applied mathematics. Especially for the computation of integrals, randomization will usually speed up the order of convergence compared to deterministic methods. It is well known that for certain function approximation problems Monte Carlo helps in a similar way. This is basically due to the fundamental work of Mathé 1991 [9] and Heinrich 1992 [7].

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Function spaces of dominating mixed smoothness were introduced by S.M. Nikol'skij in the early 1960s. Recently, there is an increasing interest in information-based complexity and high-dimensional approximation in these spaces. Function spaces of this type also play an important role in many real-world problems. For example, there exist a number of problems in finance and quantum chemistry modelled on function spaces of dominating mixed smoothness. We refer to the monographs [6,17].

Let $\mathbb{T}^d := [0,1)^d$ be the d-dimensional torus and $\mathbf{W}^r_p(\mathbb{T}^d)$ be the periodic Sobolev spaces of dominating mixed smoothness $r \geq 0$ and integrability $1 < p < \infty$ on $\mathbb{T}^d$. In this paper we study L_∞-approximation of the class $\mathbf{W}^r_p(\mathbb{T}^d)$ where we supplement the results of Fang and Duan [4,5] on L_q-approximation. Let us mention that the study of L_∞-approximation of function spaces with dominating mixed smoothness is much harder compared to the case $1 < q < \infty$ and different techniques are needed, see comments and open problems in [3, Section 4.6]. Moreover, we hope that the way we present the algorithm here will illuminate the nature of randomized approximation via linear information towards a better understanding of general L_q-approximation as well.

Let $e^{\mathrm{det,lin}}(n,S)$ and $e^{\mathrm{ran,lin}}(n,S)$ denote the minimal deterministic and randomized errors for linear approximation of the operator S if we use n deterministic or randomized information operations from the class of all linear functionals, respectively. By using an estimate on the expected norm of random trigonometric polynomials we can bound the order of convergence for linear Monte Carlo approximation,

$$\left(\frac{(\log n)^{(d-1)}}{n}\right)^{r-(\frac{1}{p}-\frac{1}{2})_+} \preceq e^{\mathrm{ran,lin}}\left(n, \mathbf{W}^r_p(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)\right) \preceq \left(\frac{(\log n)^{(d-1)}}{n}\right)^{r-(\frac{1}{p}-\frac{1}{2})_+} \sqrt{\log n}$$

with $r > \max\{\frac{1}{p}, \frac{1}{2}\}$. Comparing our result with the already known result on deterministic approximation,

$$e^{\mathrm{det,lin}}\left(n, \mathbf{W}^r_2(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)\right) \asymp \frac{(\log n)^{r(d-1)}}{n^{r-1/2}}, \qquad \text{if } r > \frac{1}{2},$$

see Temlyakov [14], we observe that for $p = 2$ randomization improves the order of convergence by a factor $n^{1/2}$, apart from a logarithmic gap. Up to this gap, our result fits to the picture obtained for random linear L_q-approximation by Fang and Duan [5], we only cite the cases with $\max\{2,p\} < q$ for which randomization does help,

$$e^{\mathrm{ran,lin}}\left(n, \mathbf{W}^r_p(\mathbb{T}^d) \hookrightarrow L_q(\mathbb{T}^d)\right) \asymp \begin{cases} (n^{-1}(\log n)^{d-1})^r & \text{if } 2 \leq p < q < \infty,\ r > \dfrac{1}{2}, \\ (n^{-1}(\log n)^{d-1})^{r-(\frac{1}{p}-\frac{1}{2})} & \text{if } p < 2 \leq q < \infty,\ r > \dfrac{1}{p}. \end{cases}$$

The study of Monte Carlo methods for L_∞-approximation is particularly interesting concerning the aspect of nonlinearity. While in the deterministic setting linear methods are always optimal, see [12, Theorem 4.5 and 4.8], in the randomized setting the situation is different. In combination with known results on nonlinear deterministic methods for L_2-approximation (compare (4.6)) we shall show that the optimal Monte Carlo approximation rate for spaces $\mathbf{W}^r_p(\mathbb{T}^d)$ with $1 < p < 2$ is better than what can be achieved with merely linear methods. More precisely, we prove

$$\left(\frac{(\log n)^{(d-1)}}{n}\right)^{r} \preceq e^{\mathrm{ran,nonlin}}\left(n, \mathbf{W}^r_p(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)\right) \preceq \left(\frac{(\log n)^{(d-1)}}{n}\right)^{r} \sqrt{\log n}$$

for $1 < p < 2$ and $r > 1$, or $2 \leq p < \infty$ and $r > 1/2$. This also consistently fits to the picture Fang and Duan [4] obtained for random L_q-approximation,

$$e^{\mathrm{ran,nonlin}}\left(n, \mathbf{W}^r_p(\mathbb{T}^d) \hookrightarrow L_q(\mathbb{T}^d)\right) \asymp (n^{-1}(\log n)^{d-1})^r, \qquad \text{if } p < q < \infty,\ r > 1.$$

The algorithm we present and analyse in Section 4 allows to reproduce the upper bounds of Fang and Duan [4,5] in a constructive way. The paper is organized as follows. In the second section we shall recall some definitions from information-based complexity and give basic properties of error notions. The next section is devoted to a fundamental Monte Carlo function approximation method in an abstract

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