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Optimization approaches to quadrature: New characterizations of Gaussian quadrature on the line and quadrature with few nodes on plane algebraic curves, on the plane and in higher dimensions[☆]

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ABSTRACT

Let d and k be positive integers. Let μ be a positive Borel measure on $\mathbb{R}^2$ possessing finite moments up to degree $2d-1$. If the support of μ is contained in an algebraic curve of degree k, then we show that there exists a quadrature rule for μ with at most dk many nodes all placed on the curve (and positive weights) that is exact on all polynomials of degree at most $2d-1$. This generalizes both Gauss and (the odd degree case of) Szegő quadrature where the curve is a line and a circle, respectively, to arbitrary plane algebraic curves. We use this result to show that, without any hypothesis on the support of μ, there is always a cubature rule for μ with at most $\frac{3}{2}d(d-1)+1$ many nodes. In both results, we show that the quadrature or cubature rule can be chosen such that its value on a certain positive definite form of degree $2d$ is minimized. We characterize the unique Gaussian quadrature rule on the line as the one that minimizes this value or several other values as for example the sum of the nodes' distances to the origin. The tools we develop should prove useful for obtaining similar results in

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higher-dimensional cases although at the present stage we can present only partial results in that direction.

1. Motivation

Initially designating the computation of areas and volumes, the terms *quadrature* and *cubature* now often stand for the numerical computation of one-dimensional and two-dimensional integrals, respectively. As a generic term for integrals over arbitrary dimension, the term *quadrature* seems to be more often used than *cubature*. We will formally use both terms synonymously but use the latter term only when the support of μ is a subset of $\mathbb{R}^2$.

By a *measure* on $\mathbb{R}^n$ we always understand a nonnegative (i.e., unsigned) Borel measure on $\mathbb{R}^n$. Its *support* is the smallest closed subset of $\mathbb{R}^n$ whose complement has measure zero. Suppose μ is a measure on $\mathbb{R}^n$ and f is a measurable real valued function whose domain contains the support of μ and whose integral with respect to μ exists and is finite. The aim is to compute the integral of f numerically, i.e., the computation should be fast and should yield a good approximation of the actual integral. Ideally, one should be able to have an error estimate for the approximation and black box access to f should be enough (in particular no information on potentially existing derivatives or primitives of f is needed).

A classical way of achieving this are *quadrature rules* [10]. They consist of finitely many points in $\mathbb{R}^n$ called *nodes* together with associated real numbers called *weights*. The hope is that the weighted sum of the function values at the nodes approximates well the integral of f with respect to μ (in the actual computation one has to deal with floating approximations, of course). It is not indispensable but highly desirable that all weights are positive since this reflects the monotonicity of the integral, increases numerical stability and usually allows for tighter error estimates [23, Conclusion 3.19].

We will therefore always insist on the weights to be positive.

Fix a nonnegative integer d such that μ possesses finite moments up to degree d, i.e., all polynomials of degree $\leq d$ have a finite integral with respect to μ. Then a quadrature rule for μ is often designed to yield the *exact value* for the integral of an arbitrary polynomial of degree at most d with respect to μ. In this way, for any function f that can be well-approximated by a polynomial of degree at most d, simultaneously

- on the support of μ or at least where "most of the mass of μ lies" (i.e., on a measurable subset of the support of μ whose complement has reasonably small measure) and
- on the nodes of the quadrature rule,

one and the same quadrature rule will give a good approximation for the integral. In practice, one often works with small degree of exactness d by splitting the domain of integration into many parts and integrating over each part separately. One thus often has a good polynomial approximation of low or moderate degree with a good error analysis, e.g., by Taylor or Bernstein approximation. Neither the subdivision (which can be adaptive to the problem) nor the error analysis is addressed in this article.

Note that one usually *would not want to, could not, need not and does not* compute a polynomial approximation of f. Thus the real aim of a quadrature rule is not to integrate polynomials which is an easy task anyway, as soon as the relevant moments of the measure are known. However, quadrature rules should be designed to handle this easy task in the best possible way. That is why this article like a big part of the literature about quadrature is about integrating *polynomials* with quadrature rules.

Of course, it is important to have a small number of nodes to speed up the computation, especially when calculating nested multiple integrals with respect to the same measure.

In addition, nodes that are far away from the origin should be evitated. Note that, due to the assumption on the finiteness of the moments up to some degree, usually most of the mass of the measure cannot lie too far from the origin and therefore it seems unlikely that the given function f could be approximated at the same time on "most of the mass of μ lies" and at "nodes that are far out". Indeed, at first sight, it seems reasonable to require that all nodes lie in the support of μ. We will however give up on this frequently made requirement for one bad and several good reasons: