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On the dispersion of sparse grids ☆

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ABSTRACT

For any $d \in \mathbb{N}$ and $\varepsilon \in (0, 1)$, we present a point set in the d-dimensional unit cube $[0, 1]^d$ that intersects every axis-aligned box of volume greater than ε. This point set is very easy to handle and in a vast range for ε and d, we do not know any smaller set with this property.

1. The result

The dispersion of a point set P in $[0, 1]^d$ is the volume of the largest axis-aligned box in $[0, 1]^d$ which does not intersect P, that is

$$\mathrm{disp}(P) = \sup \left\{ |B| : B \subset [0, 1]^d \text{ box with } B \cap P = \emptyset \right\} .$$

Here, a set $B \subset \mathbb{R}^d$ is called a box if it is the Cartesian product of d open intervals. Its volume $|B|$ is the product of the interval lengths. Point sets with small dispersion already proved to be useful for the uniform recovery of rank one tensors [4] and for the discretization of the uniform norm of trigonometric polynomials [9]. Recently, great progress has been made in the question for the minimal size for which there *exists* a point set whose dispersion is at most ε, see Dumitrescu and Jiang [3], Aistleitner, Hinrichs and Rudolf [1], Rudolf [7], Sosnovec [8] and Ullrich and Vybíral [10]. In this note, we shall *provide* a small point set that achieves the desired dispersion. This point set has a simple geometric structure. It is generated by the one-dimensional sets

$$M_j = \left\{ \frac{1}{2^{j+1}}, \frac{3}{2^{j+1}}, \ldots, \frac{2^{j+1}-1}{2^{j+1}} \right\}$$

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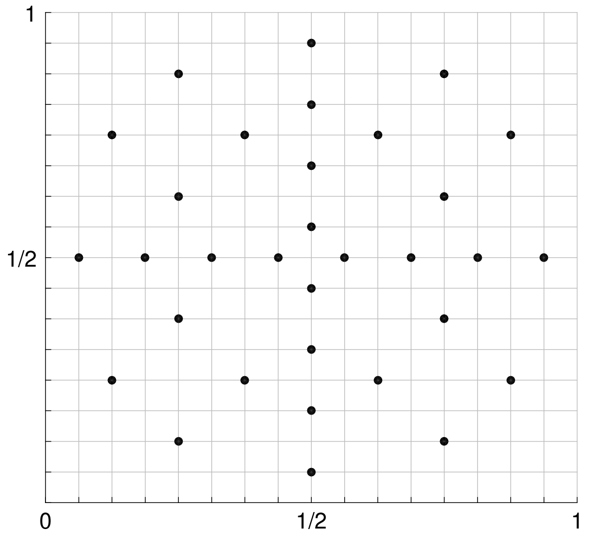

Fig. 1. The point set $P(3,2)$. This picture shows the set $P(k,d)$ of order 3 in dimension 2. The largest empty box has the volume $1/16$, the size of 16 of the little squares. If any of the 32 points is removed, an empty box of volume $1/8$ emerges.

for $j \in \mathbb{N}_0$. The d-dimensional point set of order $k \in \mathbb{N}_0$ is defined as

$$P(k,d) = \bigcup_{|\boldsymbol{j}|=k} M_{j_1} \times \cdots \times M_{j_d},$$

where we write $|\boldsymbol{j}| = j_1 + \ldots + j_d$ for $\boldsymbol{j} = (j_\ell)_{\ell=1}^d$ in $\mathbb{N}_0^d$. A picture of the set of order 3 in dimension 2 can be found in Fig. 1. These point sets are particular instances of a sparse grid as widely used for high-dimensional numerical integration and approximation. We refer to Novak and Woźniakowski [5] and the references therein. Here, we will prove the following result.

Theorem. *Let $\varepsilon \in (0,1)$ and $d \geq 2$. If we choose $k(\varepsilon) = \lceil \log_2(\varepsilon^{-1}) \rceil - 1$, the dispersion of the set $P(k(\varepsilon), d)$ is at most ε and its size is given by*

$$|P(k(\varepsilon),d)| = 2^{k(\varepsilon)} \binom{d+k(\varepsilon)-1}{d-1}.$$

Note that $|P|$ refers to the number of elements of P, if the set P is finite. The formula for $|P(k(\varepsilon),d)|$ may be simplified. On the one hand, we have

$$|P(k(\varepsilon),d)| \leq \varepsilon^{-1} \lceil \log_2(\varepsilon^{-1}) \rceil^{d-1},$$

which shows that the size roughly grows linearly in $1/\varepsilon$ for a fixed dimension d. On the other hand,

$$|P(k(\varepsilon),d)| \leq (2d)^{k(\varepsilon)},$$

which shows that the size grows at most polynomially in d for a fixed error tolerance ε. Although very simple, $P(k(\varepsilon),d)$ is the smallest explicitly known point set in $[0,1]^d$ with dispersion at most ε for many instances of ε and d, see Section 3.

2. The proof

In the following, we write $[d] = \{1,\ldots,d\}$ for each $d \in \mathbb{N}$. The vector $\boldsymbol{e}_\ell \in \mathbb{R}^d$ has entry 1 in the ℓth and 0 in all other coordinates. We start with computing the number of elements in $P(k,d)$ for $k \in \mathbb{N}_0$ and $d \in \mathbb{N}$.