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An upper bound on the minimal dispersion☆

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ABSTRACT

For $\varepsilon \in (0, 1/2)$ and a natural number $d \geq 2$, let N be a natural number with

$$N \geq 2^9 \log_2(d) \left(\frac{\log_2(1/\varepsilon)}{\varepsilon}\right)^2.$$

We prove that there is a set of N points in the unit cube $[0, 1]^d$, which intersects all axis-parallel boxes with volume ε. That is, the dispersion of this point set is bounded from above by ε.

1. Introduction

We are interested in bounds on the volume of the largest axis-parallel box that does not contain any point from a given finite point set $\mathcal{P} \subset [0, 1]^d$. Moreover, we would like to find a point set such that this volume is as small as possible. To be precise, we define, for $d \in \mathbb{N}$ and a point set $\mathcal{P} \subset [0, 1]^d$, the *dispersion of* $\mathcal{P}$ by

$$\operatorname{disp}(\mathcal{P}) := \sup_{B: B \cap \mathcal{P} = \emptyset} |B|,$$

where the supremum is over all axis-parallel boxes $B = I_1 \times \cdots \times I_d$ with intervals $I_\ell \subset [0, 1]$, and $|B|$ denotes the (Lebesgue) volume of B. Moreover, for $n, d \in \mathbb{N}$, let the *nth-minimal dispersion* be defined by

$$\operatorname{disp}(n, d) := \inf_{\substack{\mathcal{P} \subset [0,1]^d: \\ \#\mathcal{P} = n}} \operatorname{disp}(\mathcal{P})$$

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and, for $\varepsilon \in (0, 1)$, define its inverse function

$$N(\varepsilon, d) \;:=\; \min\Big\{n : \mathrm{disp}(n, d) \le \varepsilon\Big\}.$$

These quantities were introduced by Rote and Tichy [14] (as a modification of a quantity considered by Hlawka [6]) and attracted quite a lot of attention in the past years in the context of information-based complexity theory, where the explicit dependence of certain geometric quantities on the dimension d plays a crucial role. Bounds on the dispersion (or any of its variants) translate into bounds on worst-case errors (and hence complexity bounds) for several numerical problems. These include optimization in different settings [9,21], approximation of high-dimensional rank-1 tensors [2,12] and, very recently, approximation of L_p-norms and Marcinkiewicz-type discretization [17–19]. However, it is still not clear so far, if there exists a numerical problem that corresponds to the dispersion in the same way as the *discrepancy* corresponds to numerical integration, see e.g. [3–5,10,11,13].

Besides this, the dispersion is clearly an interesting geometric quantity on its own. It is easy to define and one might think it is also simple to tackle. But, as the dispersion still resists a precise analysis, this does not seem to be the case. However, there are several upper and lower bounds on the minimal dispersion, most of which were established in the past three years. Here we comment briefly on the state of the art.

First of all, it is quite easy to see that the minimal dispersion is of order n^{-1} for all d. The best bounds of this order so far, which show also an explicit dependence on d, are

$$\frac{\log_2(d)}{4(n+\log_2(d))} \;\le\; \mathrm{disp}(n, d) \;\le\; \frac{C^d}{n}$$

for some constant $C < \infty$. The lower bound is due to Aistleitner et al. [1] and the upper bound was obtained by Larcher [8] (see [1, Section 4] for the proof). Concerning the dependence on the dimension d, we see that the above bounds are far from being tight. However, it was recently proved by Sosnovec [16], that (surprisingly) the logarithmic dependence in the lower bound is sharp. He proved that, for every fixed $\varepsilon > 0$,

$$N(\varepsilon, d) \;\le\; c_\varepsilon \log_2(d).$$

However, in this bound the ε-dependence is far off. Further results on the dispersion are polynomial (in d and $1/\varepsilon$) bounds by Rudolf [15] (see Remark 1) and an explicit construction based on sparse grids by Krieg [7]. Interestingly, a lower bound linear in d was recently obtained by one of the authors [20] in the periodic setting.

It seems reasonable to conjecture that $\mathrm{disp}(n, d) \asymp \log_2(d)/n$. However, it is not yet clear if this bound can hold for all n and d.

In this article we refine the analysis of [16] paying attention to the ε-dependence and narrow the existing gap. We prove an upper bound on the inverse of the minimal dispersion that is logarithmic in d and almost quadratic in $1/\varepsilon$.

Theorem 1. *Let $d \ge 2$ be a natural number and $\varepsilon \in (0, 1/2)$. Then there exists a point set $\mathcal{P} \subset [0, 1]^d$ with* $\mathrm{disp}(\mathcal{P}) \le \varepsilon$ *and*

$$\#\mathcal{P} \;\le\; 2^7 \log_2(d)\, \frac{\big(1+\log_2(\varepsilon^{-1})\big)^2}{\varepsilon^2}.$$

Clearly, the right hand side is bounded above by the N given in the abstract. Moreover, Theorem 1 directly implies the following.

Corollary 2. *For $n, d \in \mathbb{N}$ with $n \ge 2$ and $d \ge 2$ we have*

$$\mathrm{disp}(n, d) \;\le\; c \log_2(n) \sqrt{\frac{\log_2(d)}{n}}$$

for some absolute constant $c > 0$.