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Tensor power sequences and the approximation of tensor product operators☆

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ABSTRACT

The approximation numbers of the L_2-embedding of mixed order Sobolev functions on the d-torus are well studied. They are given as the nonincreasing rearrangement of the dth tensor power of the approximation number sequence in the univariate case. I present results on the asymptotic and preasymptotic behavior for tensor powers of arbitrary sequences of polynomial decay. This can be used to study the approximation numbers of many other tensor product operators, like the embedding of mixed order Sobolev functions on the d-cube into $L_2\left([0,1]^d\right)$ or the embedding of mixed order Jacobi functions on the d-cube into $L_2\left([0,1]^d, w_d\right)$ with Jacobi weight w_d.

1. Introduction and results

Let $\sigma : \mathbb{N} \to \mathbb{R}$ be a nonincreasing zero sequence. For any natural number d, its dth tensor power is the sequence $\sigma_d : \mathbb{N}^d \to \mathbb{R}$, where

$$\sigma_d(n_1, \ldots, n_d) = \prod_{j=1}^{d} \sigma(n_j). \tag{1.1}$$

Any such sequence σ_d can then be uniquely rearranged to a nonincreasing zero sequence $\tau : \mathbb{N} \to \mathbb{R}$. Tensor power sequences like this occur naturally in the study of approximation numbers of tensor power operators. If σ is the sequence of approximation numbers of a compact operator between two

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Hilbert spaces, then τ is the sequence of approximation numbers of the compact dth tensor power operator between the tensor power spaces.

What can we say about the behavior of τ based on the behavior of σ? A classical result of Babenko [1] and Mityagin [10] is concerned with the speed of decay of these sequences:

Theorem 1. *Let σ be a nonincreasing zero sequence and τ be the nonincreasing rearrangement of its dth tensor power. For any $s > 0$, the following holds.*

(i) *If $\sigma(n) \preccurlyeq n^{-s}$, then $\tau(n) \preccurlyeq n^{-s}(\log n)^{s(d-1)}$.*
(ii) *If $\sigma(n) \succcurlyeq n^{-s}$, then $\tau(n) \succcurlyeq n^{-s}(\log n)^{s(d-1)}$.*

Here, the symbol $\preccurlyeq$ (respectively $\succcurlyeq$) means that the left (right) hand side is bounded above by a constant multiple of the right (left) hand side for all $n \in \mathbb{N}$. Of course, other decay assumptions on σ may be of interest. For instance, Pietsch [14] and König [7] study the decay of τ, if σ lies in the Lorentz sequence space $\ell_{p,q}$ for positive indices p and q, which is a stronger assumption than (i) for $s = 1/p$ but weaker than (i) for any $s > 1/p$. However, since we are motivated by the example of Sobolev embeddings, we will stick to the assumptions of Theorem 1. One of the problems with this theorem is that it does not provide explicit estimates for $\tau(n)$, even if n is huge. This is because of the constants hidden in the notation. But Theorem 1 can be sharpened.

Theorem 2. *Let σ be a nonincreasing zero sequence and τ be the nonincreasing rearrangement of its dth tensor power. For $c > 0$ and $s > 0$, the following holds.*

(i) *If $\sigma(n) \lesssim c\, n^{-s}$, then $\tau(n) \lesssim \frac{c^d}{(d-1)!^s}\, n^{-s}(\log n)^{s(d-1)}$.*
(ii) *If $\sigma(n) \gtrsim c\, n^{-s}$, then $\tau(n) \gtrsim \frac{c^d}{(d-1)!^s}\, n^{-s}(\log n)^{s(d-1)}$.*

We write $f(n) \lesssim g(n)$ for positive sequences f and g and say that $f(n)$ is asymptotically smaller or equal than $g(n)$, if the limit superior of $f(n)/g(n)$ is at most one as n tends to infinity. Analogously, $f(n)$ is asymptotically greater than or equal to $g(n)$, write $f(n) \gtrsim g(n)$, if the limit inferior of this ratio is at least one. Finally, we say $f(n)$ is asymptotically equal to $g(n)$ and write $f(n) \simeq g(n)$ if the limit of the ratio equals one. In particular, we obtain that $\sigma(n) \simeq c\, n^{-s}$ implies that $\tau(n) \simeq \frac{c^d}{(d-1)!^s}\, n^{-s}(\log n)^{s(d-1)}$. Theorem 2 is due to Theorem 4.3 in [9]. There, Kühn, Sickel and Ullrich prove this asymptotic equality in an interesting special case: τ is the sequence of approximation numbers for the L_2-embedding of the tensor power space $H^s_{\text{mix}}\left(\mathbb{T}^d\right)$ on the d-torus $[0, 2\pi]^d$, equipped with a tensor product norm. The statement can be deduced from this special case with the help of their Lemma 4.14. However, we prefer to give a direct proof in Section 2 by generalizing the proof of Theorem 4.3 in [9].

Theorem 2 gives us a pretty good understanding of the asymptotic behavior of the dth tensor power τ of a sequence σ of polynomial decay. If $\sigma(n)$ is roughly $c\, n^{-s}$ for large n, then $\tau(n)$ is roughly $c^d \left(\frac{(\log n)^{d-1}}{(d-1)!}\right)^s n^{-s}$ for n larger than a certain threshold. But even for modest values of d, the size of this threshold may go far beyond the scope of computational capabilities. Indeed, while τ decreases, the function $n^{-s}(\log n)^{s(d-1)}$ grows rapidly as n goes from 1 to e^{d-1}. For $n^{-s}(\log n)^{s(d-1)}$ to become less than one, n even has to be super exponentially large in d. Thus, any estimate for the sequence τ in terms of $n^{-s}(\log n)^{s(d-1)}$ is useless to describe its behavior in the range $n \leq 2^d$, its so called preasymptotic behavior. As a replacement, we will prove the following estimate in Section 3.

Theorem 3. *Let σ be a nonincreasing zero sequence and τ be the nonincreasing rearrangement of its dth tensor power. Let $\sigma(1) > \sigma(2) > 0$ and assume that $\sigma(n) \leq C\, n^{-s}$ for some $s, C > 0$ and all $n \geq 2$. For any $n \in \left\{2, \ldots, 2^d\right\}$,*

$$\frac{\sigma(2)}{\sigma(1)} \cdot \left(\frac{1}{n}\right)^{\frac{\log(\sigma(1)/\sigma(2))}{\log\left(1+\frac{d}{\log_2 n}\right)}} \leq \frac{\tau(n)}{\tau(1)} \leq \left(\frac{\exp\left(\left(C/\sigma(1)\right)^{2/s}\right)}{n}\right)^{\frac{\log(\sigma(1)/\sigma(2))}{\log\left(\left(\sigma(1)/\sigma(2)\right)^{2/s} d\right)}}.$$

Let us assume the power (or dimension) d to be large. Then the tensor power sequence, which roughly decays like n^{-s} for huge values of n, roughly decays like n^{-t_d} with $t_d = \log\left(\sigma(1)/\sigma(2)\right)/\log d$

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