ARTICLE IN PRESS YJDEQ:9230

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Journal of Differential Equations

www.elsevier.com/locate/jde

Minimal graphs with micro-oscillations

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Received 16 March 2016

Abstract

We show that there are minimal graphs in $\mathbb{R}^{n+1}$ whose intersection with the portion of the horizontal hyperplane contained in the unit ball has any prescribed geometry, up to a small deformation. The proof hinges on the construction of minimal graphs that are almost flat but have small oscillations whose geometry we can control.

1. Introduction

Let us consider minimal graphs on the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$. More precisely, let u be a function satisfying the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0 \tag{1}$$

in $\mathbb{B}^n$. This is equivalent to saying that the graph of u,

$$\Sigma_u := \left\{(x, u(x)) : x \in \mathbb{B}^n\right\}$$

is a minimal hypersurface of $\mathbb{R}^{n+1}$.

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https://doi.org/10.1016/j.jde.2018.02.025

Area bounds for minimal graphs play a key role in the theory of minimal surfaces. Since Σ_u is a global minimizer of the area functional among the surfaces with fixed boundary, if u is bounded on $\mathbb{B}^n$ it is clear that the n-dimensional Hausdorff measure $\mathcal{H}^n(\Sigma_u)$ is bounded by a non-uniform constant that depends on the oscillation of u, $\operatorname{osc} u := \max_{\mathbb{B}^n} u - \min_{\mathbb{B}^n} u$. On the contrary, if we just specify the value of the oscillation of u, we can consider a hyperplane of slope $\frac{\operatorname{osc} u}{2}$, which is a minimal hypersurface whose area $\mathcal{H}^n(\Sigma_u)$ is bounded from below by a constant increasing as $\operatorname{osc} u$. Therefore, there is not a uniform estimate holding for the area of minimal graphs on $\mathbb{B}^n$.

Our objective in this note is to explore the non existence of a higher-codimension analog of the uniform estimate. We will be interested in bounds for the $(n-1)$-dimensional Hausdorff measure of the hypersurface, or rather of its *transverse* intersection with a hyperplane. To this end, let us denote by Π the portion of the horizontal hyperplane that is contained in the unit ball of $\mathbb{R}^{n+1}$:

$$\Pi := \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\,,\ |x| < 1\}\,.$$

Actually, our goal is to show a more general result claiming that we can prescribe the geometry of the intersection of a minimal graph on $\mathbb{B}^n$ with Π, up to a small deformation:

Theorem 1. *Let S be a compact, connected, properly embedded, orientable hypersurface of $\mathbb{B}^n$ with nonempty boundary. Then, for any integer k and $\epsilon > 0$, there is a minimal graph over the unit n-ball and an open subset $\Pi' \subset \Pi$ such that the intersection $\Sigma_u \cap \overline{\Pi'}$ is given by $\Phi(S)$, where $\Phi : \Pi \to \Pi$ is a diffeomorphism with $\|\Phi - \mathrm{id}\|_{C^k} < \epsilon$.*

If one chooses S to be a compact hypersurface of $\mathbb{B}^n$ with area $\mathcal{H}^{n-1}(S) > c$ and ϵ is small enough, the immediate corollary is a codimension 1 analog of the existence of minimal graphs with arbitrarily large area:

Theorem 2. *The $(n-1)$-dimensional measure of the transverse intersection of a minimal graph over the unit n-ball with a hyperplane is not uniformly bounded. Specifically, given any constant c, there is some u satisfying the Equation (1) for which Σ_u and Π intersect transversally but*

$$\mathcal{H}^{n-1}(\Sigma_u \cap \Pi) > c\,.$$

Of course, the reason for which transverse intersections are considered is that for $u := 0$ (that is, $\Sigma_u = \Pi$) one trivially has $\mathcal{H}^{n-1}(\Sigma_u \cap \Pi) = \infty$. This fact strongly suggests that Theorem 2 should hold for graphs that are a small perturbation of the hyperplane, but the passage from this heuristic argument to an actual proof is nontrivial.

These minimal graphs with micro-oscillations play the opposite role with respect to area bounds that hyperplanes. Even hyperplanes with arbitrarily large n-measure do not have larger $(n-1)$-measure of its intersection with Π than the diameter of the ball. Our construction of minimal graphs leads to arbitrarily large $(n-1)$-measure of the transverse intersection but with n-measure less than twice the area of the ball.

The key point of Theorem 1 is that it can be analyzed in the linear regime of the minimal surface equation. In fact, the strategy that we have used to prove it (see Section 2) is to construct harmonic functions v on the ball that are small in a C^k norm and whose zero set $v^{-1}(0)$ contains

Please cite this article in press as: A. Enciso et al., Minimal graphs with micro-oscillations, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.02.025