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Long-time asymptotics for the short pulse equation

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Abstract

In this paper, we analyze the long-time behavior of the solution of the initial value problem (IVP) for the short pulse (SP) equation. As the SP equation is a completely integrable system, which posses a Wadati–Konno–Ichikawa (WKI)-type Lax pair, we formulate a 2×2 matrix Riemann–Hilbert problem to this IVP by using the inverse scattering method. Since the spectral variable k is the same order in the WKI-type Lax pair, we construct the solution of this IVP parametrically in the new scale (y, t), whereas the original scale (x, t) is given in terms of functions in the new scale, in terms of the solution of this Riemann–Hilbert problem. However, by employing the nonlinear steepest descent method of Deift and Zhou for oscillatory Riemann–Hilbert problems, we can get the explicit leading order asymptotic of the solution of the short pulse equation in the original scale (x, t) as time t goes to infinity.

Keywords: Riemann–Hilbert problem; Short pulse equation; Initial value problem; Long-time asymptotics

1. Introduction

The present work is devoted to the study of the long-time asymptotic behavior of the short pulse (SP) equation formulated on the whole line,

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1a}$$

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where $u(x,t)$ is a real-valued function, which represents the magnitude of the electric field, while the subscripts t and x denote partial differentiations, with the initial value data

$$u(x,t=0)=u_0(x), \quad x\in\mathbb{R}, \tag{1.1b}$$

and assuming that $u_0(x)$ lies in Schwartz space.

The SP equation was proposed in [1] by Schäfer and Wayne to describe the propagation of ultra-short optical pulses in silica optical fibers. Usually, in nonlinear optics, the nonlinear Schrödinger (NLS) equation was always used to model the slowly varying wave trains. As the pulse duration shortens, however, the NLS equation becomes less accurate, the SP equation provides an increasingly better approximation to the corresponding solution of the Maxwell equations [2]. For the details of physical background, see [1] and references therein.

Actually, the SP equation appeared first as one of Rabelo's equations which describe pseudospherical surfaces, possessing a zero-curvature representation, in [3]. Recently, the Wadati–Konno–Ichikawa (WKI) type Lax pair of the SP equation was rediscovered in [4] (see the following (2.1a)). The integrable properties of SP equation like bi-Hamiltonian structure and the conservation laws were studied in [5,6]. The loop-soliton solutions the short pulse equation was found in [7]. The connection between the short pulse equation and the sine-Gordon equation through the hodograph transformation was found by Matsuno, and thus, multi-soliton solutions including multi-loop and multi-breather ones were given in [8]. And a lot of generalizations of the SP equation, such as vector SP equation, discretizations of SP equation, complex SP equation,and so on, were studied in [9–11] and references therein.

The local well-posedness in H^2 (which denotes the usual Sobolev space) and non-existence of smooth traveling wave solutions were shown in [1], and global well-posedness of small solutions was proved in [12] for SP equation in H^2 by using conservation laws. In [13], Liu, Pelinovsky and Sakovich showed the blow-up result for the SP equation for large data.

The purpose of this paper is to analyze the long-time asymptotic behavior of the SP equation. Due to the SP equation admits a Lax pair, the inverse scattering transform method can be used to solve the initial value problem for the SP equation. Here, we relate the inverse scattering problem to a 2×2-matrix Riemann–Hilbert problem. The most important advantage of formulating the initial value problem (1.1a)–(1.1b) as a Riemann–Hilbert problem is that the long-time asymptotic behavior of the solution of the initial value problem can be analyzed by employing the nonlinear steepest descent method introduced by Deift and Zhou [14]. This method has previous applied to many integrable equations, such as the NLS equation [15], the Sine–Gordon equation [16], the KdV equation [17], the Fokas–Lenells equation [18], the Camassa–Holm equation [19] and so on.

Recently, this approach has been applied to the so-called short-wave approximations of integrable equations, which themselves are integrable, such as the modified Hunter–Saxton (mHS) equation [20] and the Ostrovsky–Vakhnenko (OV) equation [21], which can be viewed as the short-wave limit of the Camassa–Holm and Degasperis–Procesi equations, respectively. The SP equation considered in this paper can be viewed as the short-wave limit of the modified Camassa–Holm equation [22] (see, also [23]). All of the short-wave limit equations named above have the common feature that their solutions can be extracted from the development of the solutions of the respective Riemann–Hilbert problems at $k\to 0$. Although the long-time asymptotic analysis of (1.1a) is in many ways similar to those of integrable equations, it also presents some distinctive features: *(1)* **The spectral variable k is the same order in the Lax pair**, it firstly has to be transformed by introduction of a matrix $G(x,t)$ (see the following equation (2.12)) to arrive

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