ARTICLE IN PRESS YJDEQ:9343

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Quadratic differential systems with complex conjugate invariant lines meeting at a finite point

Dana Schlomiuk [a,*], Xiang Zhang [b]

[a] *Département de Mathématiques et de Statistiques, Université de Montréal, CP 6128 Succ. A, Montreal PQ, H3C 3J7, Canada*

[b] *School of Mathematical Sciences, Shanghai Jiaotong University, Shanghai, 200240, PR China*

Received 6 March 2018; revised 9 May 2018

Abstract

In this article we study the class QS2cIL of quadratic differential systems with two complex conjugate invariant lines meeting at a finite point. From the literature we know that quadratic systems with invariant lines of total multiplicity at least four or with the line at infinity filled up with singularities are integrable via the method of Darboux and hence they have no limit cycles. These could only occur if we have only the two complex lines, and the line at infinity, all simple. We first find all integrable systems in QS2cIL due to the presence of invariant lines. We next indicate a gap in the 1986 proof of Suo and Chen that systems in QS2cIL have at most one limit cycle and we give a complete proof of this result. Finally we give the topological classification of QS2cIL yielding 22 phase portraits three of which with a limit cycle.

MSC: 58K45; 34C37; 37C10; 37C29; 37J45; 70H05

Keywords: Quadratic vector fields; Configuration of invariant lines; Darboux integrability; Limit cycle

* Corresponding author.
E-mail addresses: dasch@DMS.UMontreal.CA (D. Schlomiuk), xzhang@sjtu.edu.cn (X. Zhang).

https://doi.org/10.1016/j.jde.2018.05.014

1. Introduction and statement of the main results

Consider differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

with p, q polynomials in x and y with real coefficients and their associated vector fields

$$\tilde{D} = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}. \tag{2}$$

We call *degree* of such a system the number $max\{\deg p, \deg q\}$. A system (1) is called *quadratic* if its degree is two. We denote by **QS** the class of all quadratic systems. A polynomial system (1) is *degenerate* if and only if the polynomials p and q have a non-constant common factor.

Quadratic differential systems, sometimes simply called quadratic systems, have been intensely studied using a mixture of algebraic, geometric, analytic and numerical methods. Over one thousand papers on these systems were published.

In recent years we have witnessed a surge of activity on quadratic and more generally on polynomial differential systems, using the theory of integrability of Darboux [8], which is of an algebraic-geometric nature. In particular, in this article, we base our study on theorems concerning the integrability of quadratic systems resulting from the presence of invariant straight lines, i.e. of lines which are unions of phase curves (see [12–16]). This is the algebraic part of the work. The analytic part concerns the study of limit cycles.

We point out here that in the articles mentioned above, classification theorems for families of quadratic systems were done in invariant form, i.e. independent of the normal form in which the systems were presented. This was achieved by using invariant polynomials. Other examples of articles presenting intrinsic classification theorems in terms of invariant polynomials are [3,17,18].

Limit cycles in **QS** could also be algebraic. The interested reader could find some examples of quadratic systems with algebraic limit cycles in [7,10].

In this paper we study the class $\mathbf{QS}_{2cIL}$ of all quadratic differential systems (1) which have a pair of complex conjugate invariant lines intersecting at a real finite point. These systems thus possess at least three invariant lines: the two complex lines and the line at infinity.

One of our aims in this paper is to give a complete *topological classification* of such systems, i.e. the classification of the systems with respect to the topological equivalence relation. By this we mean that two systems (1) are *topologically equivalent* if and only if there exists a homeomorphism of the plane carrying orbits to orbits and preserving their orientation. However, in order to half the number of phase portraits, we use a coarser equivalence relation, namely we say that two systems (1) are *topologically equivalent* if and only if there exists a homeomorphism of the plane carrying orbits to orbits and preserving or reversing their orientation.

As mentioned above, all systems in the family $\mathbf{QS}_{2cIL}$ have at least three invariant lines, i.e. the two complex lines intersecting at a finite real point and the line at infinity. These lines could sometimes be multiple lines. For the notions of multiple invariant line and total multiplicity of invariant lines see [12,14] but since these notions are important here we also define them in our Section 3. The basic theory for invariant lines of quadratic differential systems is found in [12]. Some of these systems could have more invariant lines and these additional invariant lines may

Please cite this article in press as: D. Schlomiuk, X. Zhang, Quadratic differential systems with complex conjugate invariant lines meeting at a finite point, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.05.014