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Small data well-posedness for derivative nonlinear Schrödinger equations

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Abstract

We study the local and global solutions of the generalized derivative nonlinear Schrödinger equation $i\partial_t u + \Delta u = P(u, \overline{u}, \partial_x u, \partial_x \overline{u})$, where each monomial in P is of degree 3 or higher, in low-regularity Sobolev spaces without using a gauge transformation. Instead, we use a solution decomposition technique introduced in [4] during the perturbative argument to deal with the loss on derivative in nonlinearity. It turns out that when each term in P contains only one derivative, the equation is locally well-posed in $H^{\frac{1}{2}}$, otherwise we have a local well-posedness in $H^{\frac{3}{2}}$. If each monomial in P is of degree 5 or higher, the solution can be extended globally. By restricting to equations to the form $i\partial_t u + \Delta u = \partial_x P(u, \overline{u})$ with the quintic nonlinearity, we were able to obtain the global well-posedness in the critical Sobolev space.

MSC: 35Q55; 35A01; 35B45

Keywords: Derivative nonlinear Schrödinger equations; Local well-posedness; Global well-posedness

1. Introduction

In this paper, we study the well-posedness of the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gDNLS) on $\mathbb{R}$.

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$$\begin{cases} i\partial_t u + \Delta u = P(u, \overline{u}, \partial_x u, \partial_x \overline{u}) \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0. \end{cases} \tag{1}$$

Here, u is a complex-valued function and $P : \mathbb{C}^4 \to \mathbb{C}$ is a polynomial of the form

$$P(z) = P(z_1, z_2, z_3, z_4) = \sum_{d \leq |\alpha| \leq l} C_\alpha z^\alpha, \tag{2}$$

and $l \geq d \geq 3$. There are several results regarding the well-posedness of this equation. In [19], Kenig, Ponce and Vega proved that the equation (1) is locally well-posed for a small initial data in $H^{\frac{7}{2}}(\mathbb{R})$. There has been some interest in the special case where $P = i\lambda |u|^k u_x$:

$$\begin{cases} i\partial_t u + \Delta u = i\lambda |u|^k u_x \\ u(x,0) = u_0 \in H^s(\mathbb{R}), s \geq s_0, \end{cases}$$

with $k \in \mathbb{R}$. Hao ([13]) proved that this equation is locally well-posed in $H^{\frac{1}{2}}(\mathbb{R})$ for $k \geq 5$, and Ambrose–Simpson ([1]) proved the result in $H^1(\mathbb{R})$ for $k \geq 2$. Recent studies show that these results can be improved. See Santos ([25]) for the local-wellposedness in $H^{\frac{1}{2}}$ when $k \geq 2$ and Hayashi–Ozawa ([14]) for the local well-posedness in H^2 when $k \geq 1$ and the global well-posedness in H^1 when $k \geq 2$.

Several studies showed that we have better results if P only consists of $\overline{u}$ and $\partial_x \overline{u}$ due to the following heuristic: if u solves the linear Schrödinger equation, then the space–time Fourier transform of $\overline{u}$ is supported away from the parabola $\{(\xi, \tau) | \tau + \xi^2 = 0\}$, leading to strong dispersive estimates. Grünrock ([12]) showed that for $P = \partial_x(\overline{u}^d)$ or $P = (\partial_x \overline{u})^d$ where $d \geq 3$, the equation (1) is locally well-posed for any $s > \frac{1}{2} - \frac{1}{d-1}$ in the former case and $s > \frac{3}{2} - \frac{1}{d-1}$ in the latter. Later, Hirayama ([16]) extended Grünrock's results for $P = \partial_x(\overline{u}^d)$ to the global well-posedness for $s \geq \frac{1}{2} - \frac{1}{d-1}$.

There are also various results for higher dimension analogues of (1)

$$\begin{cases} i\partial_t u + \Delta u = P(u, \overline{u}, \nabla u, \nabla \overline{u}) \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3}$$

The most general results in $\mathbb{R}^n$ for $n \geq 2$ are due to Kenig, Ponce and Vega in [19]. For a more specific case, we refer to [2] and [3] where Bejenaru obtained a local well-posedness result for $n = 2$ and $P(z)$ is quadratic with low regularity initial data. For results in Besov spaces, see [30] for the global well-posedness in $\dot{B}^{s_n}_{1,2}(\mathbb{R}^n)$ where $n \geq 2$ and $s_n = \frac{n}{2} - \frac{1}{d-1}$ which is the critical exponent.

For another type of derivative nonlinearities, we refer to Chihara ([10]) for nonlinearities of the form $f(u, \partial u)$, where $f : \mathbb{R}^2 \times \mathbb{R}^{2n} \to \mathbb{R}$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$) is a smooth function such that $f(u, v) = O(|u|^2 + |v|^2)$ or $f(u, v) = O(|u|^3 + |v|^3)$ near $(u, v) = 0$. It turns out that the corresponding Cauchy problems are locally well-posed in $H^{\lfloor n/2 \rfloor + 4}$ for any $n \geq 1$.

Our first result is the local well-posedness of (1) in Sobolev spaces when the nonlinearity contains an arbitrary number of derivatives.

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