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Discontinuous traveling waves as weak solutions to the Fornberg–Whitham equation

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Abstract

We analyze the weak solution concept for the Fornberg–Whitham equation in case of traveling waves with a piecewise smooth profile function. The existence of discontinuous weak traveling wave solutions is shown by means of analysis of a corresponding planar dynamical system and appropriate patching of disconnected orbits.

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1. Basic concepts

1.1. Introduction

The Fornberg–Whitham equation has been introduced as one of the simplest shallow water wave models which are still capable of incorporating wave breaking (cf. [4,6,7,10,12–14]). The wave height is described by a function of space and time $u\colon \mathbb{R} \times [0, \infty[\to \mathbb{R}$, $(x, t) \mapsto u(x, t)$, we will occasionally write $u(t)$ to denote the function $x \mapsto u(x, t)$. Suppose that an initial wave profile u_0 is given as a real-valued function on $\mathbb{R}$. The Cauchy problem for the Fornberg–Whitham equation is

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$$u_t + uu_x + K * u_x = 0, \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

where the convolution is in the x-variable only and

$$K(x) = \frac{1}{2}e^{-|x|}, \tag{3}$$

which satisfies $(1-\partial_x^2)K = \delta$.

We note that formally applying $1-\partial_x^2$ to (1) produces a third order partial differential equation

$$u_t - u_{txx} - 3u_x u_{xx} - uu_{xxx} + uu_x + u_x = 0,$$

but we will stay with the above non-local equation which corresponds to the original model and is also more suitable for the weak solution concept.

Remark 1.1. Note that we follow here in (1) the sign convention for the convolution term as used in [6, Equation (4)] (or also in [14, Section 13.14]), but used a rescaling of the solution by 3/2 to get rid of an additional constant factor in the nonlinear term.

Well-posedness results on short time intervals for (1)–(2) with spatial regularity according to Sobolev or Besov scales have been obtained in [8,9]. For example, in terms of Sobolev spaces these read as follows: If $s > 3/2$ and $u_0 \in H^s(\mathbb{R})$, then there exists $T_0 > 0$ such that (1)–(2) possesses a unique solution $u \in C([0,T_0], H^s(\mathbb{R})) \cap C^1([0,T_0], H^{s-1}(\mathbb{R}))$; moreover, the map $u_0 \mapsto u$ is continuous $H^s(\mathbb{R}) \to C([0,T_0], H^s(\mathbb{R}))$ and $\sup_{t\in[0,T_0]} \|u(t)\|_{H^s(\mathbb{R})} < \infty$.

1.2. Weak solution concept

Equation (1) can formally be rewritten in the form

$$\partial_t u + \partial_x\Big(\frac{u^2}{2}\Big) + K' * u = 0, \tag{4}$$

which suggests to define weak solutions in the context of locally bounded measurable functions in the following way.

Definition 1.2. A function $u \in L^\infty_{\text{loc}}(\mathbb{R}\times[0,\infty[)$ is called a *weak solution* of the Cauchy problem (1)–(2) with initial value $u_0 \in L^\infty_{\text{loc}}(\mathbb{R})$, if

$$\int_0^\infty \int_{-\infty}^\infty \Big(-u(x,t)\partial_t\phi(x,t) - \frac{u^2(x,t)}{2}\partial_x\phi(x,t) + \big(K' * u(.,t)\big)(x)\phi(x,t)\Big)\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^\infty u_0(x)\phi(x,0)\,\mathrm{d}x \tag{5}$$

holds for every test function $\phi \in \mathcal{D}(\mathbb{R}^2)$.