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Singular extremal solutions for supercritical elliptic equations in a ball

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Abstract

We study positive singular solutions to the Dirichlet problem for the semilinear elliptic equation $\Delta u + \lambda f(u) = 0$ in the unit ball on $\mathbf{R}^N$. We assume that f has the form $f(u) = u^p + g(u)$, where $p > (N+2)/(N-2)$ and $g(u)$ is a lower order term. We first show the uniqueness of the singular solution to the problem, and then study the existence of the singular extremal solution. In particular, we show a necessary and sufficient condition for the existence of the singular extremal solution in terms of the weak eigenvalue of the linearized problem.

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Keywords: Semilinear elliptic equation; Singular extremal solution; Joseph–Lundgren exponent; Uniqueness; Bifurcation diagram; Generalized eigenvalue

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1. Introduction and main results

We study singular solutions to the semilinear elliptic Dirichlet problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $B = \{x \in \mathbf{R}^N : |x| < 1\}$ with $N \geq 3$ and λ is a nonnegative constant. In (1.1) we assume that f has the form

$$f(u) = u^p + g(u) \quad \text{for } u \geq 0, \tag{1.2}$$

where $p > p_S := (N+2)/(N-2)$ and $g(u)$ is a lower order term. By the symmetry result of Gidas–Ni–Nirenberg [16], every regular positive solution u is radially symmetric and $\|u\|_{L^\infty} = u(0)$. It is known that the set of the solutions can be parametrized by the L^∞-norm of the solution. Let (λ, u) be a solution of (1.1) and let $\alpha = u(0) = \|u\|_{L^\infty}$. Then λ becomes a graph of α, i.e., $\lambda(\alpha)$ (see, e.g., [20]).

We recall some results about bifurcation diagrams of supercritical elliptic equations. Joseph–Lundgren [19] studied the Dirichlet problem

$$\begin{cases} \Delta u + \lambda (u+1)^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.3}$$

For $p > N/(N-2)$, we find the explicit singular solution (λ_0^*, u_0^*) with

$$\lambda_0^* = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right) \quad \text{and} \quad u_0^*(x) = |x|^{-2/(p-1)} - 1. \tag{1.4}$$

Define the exponent p_{JL} by

$$p_{JL} := \begin{cases} 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geq 11, \\ \infty, & 2 \leq N \leq 10, \end{cases}$$

which is called the Joseph–Lundgren exponent introduced in [19]. It was shown by [19] that, when $p_S < p < p_{JL}$, $\lambda(\alpha)$ oscillates infinitely many times around λ_0^* and converges to λ_0^* as $\alpha \to \infty$, and that, when $N \geq 11$ and $p \geq p_{JL}$, $\lambda(\alpha)$ is strictly increasing and it converges to λ_0^* as $\alpha \to \infty$.

The study of the problem

$$\begin{cases} \Delta u + \lambda u + u^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.5}$$

was initiated by Brezis–Nirenberg [2] in the critical case $p = p_S$. Later, the supercritical case $p > p_S$ was studied by Budd–Norbury [4], Budd [5], Merle–Peletier [23], Dolbeault–Flores [14],