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Refined estimates for the propagation speed of the transition solution to a free boundary problem with a nonlinearity of combustion type ☆

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Abstract

We are concerned with the nonlinear problem $u_t = u_{xx} + f(u)$, where f is of combustion type, coupled with the Stefan-type free boundary $h(t)$. According to [4,5], for some critical initial data, the transition solution u locally uniformly converges to θ, which is the ignition temperature of f, and the free boundary satisfies $h(t) = C\sqrt{t} + o(1)\sqrt{t}$ for some positive constant C and all large time t. In this paper, making use of two different approaches, we establish more accurate upper and lower bound estimates on $h(t)$ for the transition solution, which suggest that the nonlinearity f can essentially influence the propagation speed.

MSC: 35K20; 35K55; 35R35

Keywords: Free boundary problem; Nonlinear diffusion equation; Combustion; Propagation speed

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1. Introduction and main results

In this paper, we consider the following free boundary problem of nonlinear diffusion equations:

$$\begin{cases} u_t = u_{xx} + f(u), & t>0,\ g(t)<x<h(t), \\ u(t,g(t)) = u(t,h(t)) = 0, & t>0, \\ g'(t) = -\mu u_x(t,g(t)), & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ -g(0)=h(0)=h_0, u(0,x)=u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.1}$$

where $x = g(t)$ and $x = h(t)$ are the expanding fronts, μ, h_0 are given positive constants. The nonlinear reaction term f is a locally Lipschitz continuous function satisfying

$$f(0)=0, \quad f(u)<0 \ \text{ for } u>1. \tag{1.2}$$

Throughout the paper, unless otherwise specified, we assume that f is of combustion type:

$$\begin{aligned} & f(u)=0 \ \text{ in } [0,\theta], \quad f(u)>0 \ \text{ in } (\theta,1), \\ & f(u)<0 \ \text{ in } (1,\infty) \end{aligned} \tag{1.3}$$

for some constant $\theta \in (0,1)$.

For any given $h_0 > 0$, the initial function $u_0(x)$ satisfies

$$u_0 \in \mathscr{X}(h_0) := \left\{ \phi \in C^2[-h_0,h_0] : \begin{array}{l} \phi(-h_0)=\phi(h_0)=0,\ \phi'(-h_0)>0 \\ \phi'(h_0)<0,\ \phi(x)>0 \text{ in } (-h_0,h_0) \end{array} \right\}. \tag{1.4}$$

It was shown by [4] that under condition (1.2), (1.1) has a unique globally defined classical solution $(u(t,x), h(t), g(t))$. In addition, $g'(t) < 0$ and $h'(t) > 0$ for $t > 0$ and therefore

$$g_\infty := \lim_{t\to\infty} g(t) \in [-\infty, -h_0), \quad h_\infty := \lim_{t\to\infty} h(t) \in (h_0, +\infty]$$

always exist.

Denote $\mathbb{R} = (-\infty, \infty)$. In [4], the authors obtained the following trichotomy result and the sharp threshold dynamics when f satisfies (1.3).

Theorem A *([4, Theorem 1.4]). Assume that f is of combustion type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then one of the following situations occurs:*

(i) **Spreading:** $(g_\infty, h_\infty) = \mathbb{R}$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \ \text{ locally uniformly in } \mathbb{R};$$