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Malliavin calculus for the stochastic Cahn–Hilliard/Allen–Cahn equation with unbounded noise diffusion

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Abstract

The stochastic partial differential equation analyzed in this work, is motivated by a simplified mesoscopic physical model for phase separation. It describes pattern formation due to adsorption and desorption mechanisms involved in surface processes, in the presence of a stochastic driving force. This equation is a combination of Cahn–Hilliard and Allen–Cahn type operators with a multiplicative, white, space–time noise of unbounded diffusion. We apply Malliavin calculus, in order to investigate the existence of a density for the stochastic solution u. In dimension one, according to the regularity result in [5], u admits continuous paths a.s. Using this property, and inspired by a method proposed in [8], we construct a modified approximating sequence for u, which properly treats the new second order Allen–Cahn operator. Under a localization argument, we prove that the Malliavin derivative of u exists locally, and that the law of u is absolutely continuous, establishing thus that a density exists.

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1. Introduction

1.1. The stochastic model

We consider the following stochastic partial differential equation which is given as a combination of Cahn–Hilliard and Allen–Cahn type equations, perturbed by a multiplicative space–time noise $\dot{W}$ with a non-linear diffusion coefficient σ

$$u_t = -\varrho\Delta\Big(\Delta u - f(u)\Big) + \Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W}, \quad t > 0,\ x \in \mathcal{D}, \tag{1.1}$$

where $\mathcal{D} \subset \mathbb{R}^d$, for $d = 1, 2, 3$, is a bounded spatial domain. Here, $f(u) = u^3 - u$ is the derivative of a double equal-well potential. The constant $\varrho > 0$ is a positive bifurcation parameter referring to an attractive potential for the related physical model, while the noise $\dot{W} = \dot{W}(x, t)$ is a space–time white noise in the sense of Walsh, [17], given as the formal derivative of a Wiener process. More specifically, $dW := W(dx, ds)$ is a d-dimensional space–time white noise, induced by the one-dimensional $(d + 1)$-parameter Wiener process W defined as $W := \big\{W(x, t) :\ t \in [0, T],\ x \in \mathcal{D}\big\}$. The noise diffusion $\sigma(u)$ has a sub-linear growth of the form

$$|\sigma(u)| \le C(1 + |u|^q),$$

for some $C > 0$ and any $q \in (0, \frac{1}{3})$.

The initial and boundary value problem for this equation, satisfies the initial condition

$$u(x, 0) = u_0(x) \text{ in } \mathcal{D},$$

and the next homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \partial\mathcal{D} \times [0, T). \tag{1.2}$$

The Cahn–Hilliard equation was initially proposed as a simple model for the description of the phase separation of a binary alloy, being in a non-equilibrium state, [10]. Cook in [11], extended the deterministic partial differential equation to a stochastic one by introducing thermal fluctuations in the form of an additive noise. There exist some interesting results in the relevant literature on existence and uniqueness of solution for the stochastic problem, as for example in [8,12], where the i.b.v.p. was posed on cubic domains, and rectangles, or on Lipschitz domains of more general topography, [4]. In [7,12,8,9], the authors considered the version of an odd polynomial nonlinearity for the potential. Moreover, in [3], the one-dimensional stochastic Cahn–Hilliard equation has been approximated by a manifold of solutions and the dynamics of the stochastic motion of the fronts were described. In [7], the effect of noise on evolving interfaces during the initial stage of phase separation was analyzed, while in [6], the singular limit of the generalized Cahn–Hilliard equation has been rigorously derived by means of the Hilbert expansion method, imitating the behavior of a stochastic model. The sharp interface limit of the Cahn–Hilliard equation with additive noise has been examined in [2]; in this case, depending on the noise strength, the chemical potential satisfies on the limit a deterministic or a stochastic