Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 265 (2018) 2375–2430

Journal of Differential Equations

www.elsevier.com/locate/jde

Carleman estimates for the parabolic transmission problem and Hölder propagation of smallness across an interface

Elisa Francini *, Sergio Vessella

Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Italy

Received 23 September 2017; revised 12 April 2018
Available online 24 April 2018

Abstract

In this paper we prove a Hölder propagation of smallness for solutions to second order parabolic equations whose general anisotropic leading coefficient has a jump at an interface. We assume that the leading coefficient is Lipschitz continuous with respect to the parabolic distance on both sides of the interface. The main effort consists in proving a local Carleman estimate for this parabolic operator.

MSC: 35K10; 35R05; 35B45

Keywords: Carleman estimate; Parabolic transmission problem; Propagation of smallness

1. Introduction

The main purpose of this paper is to study unique continuation properties and propagation of smallness for solutions to second order parabolic equations whose anisotropic leading coefficients have jumps at an interface. Although there exist good general books and papers about Carleman estimates, unique continuation properties and related propagation of smallness (see [8], [14], [27], [29], [34], [59], [65]) and a lot of surveys and introductory papers on the subject and on its several applications (see [26], [31], [32], [30], [35], [42], [61], [62], [64]), we

* Corresponding author.
E-mail addresses: elisa.francini@unifi.it (E. Francini), sergio.vessella@unifi.it (S. Vessella).

https://doi.org/10.1016/j.jde.2018.04.033

would like to give to the non expert reader some basic notions and quick historic panorama on the subject.

We say that a linear partial differential equation $\mathcal{L}(u) = 0$, enjoys a unique continuation property (UCP) in a connected open set $\Omega \subset \mathbb{R}^N$ if the following property holds true [58]: for any open subset $\widetilde{\Omega}$ of Ω

$$\mathcal{L}(u) = 0 \text{ in } \Omega \text{ and } u = 0 \text{ in } \widetilde{\Omega} \quad \text{imply} \quad u = 0 \text{ in } \Omega. \tag{1.1}$$

We call quantitative estimate of unique continuation (QEUC) or stability estimate related to the UCP property (1.1) the following type of result:

$$\mathcal{L}(u) = 0 \text{ in } \Omega \text{ and } u \text{ small in } \widetilde{\Omega} \quad \text{imply} \quad u \text{ small in } \Omega.$$

Of course, the research in these topic is of some interest if either the function u is not analytic or if the operator $\mathcal{L}$ has nonanalytic coefficients in Ω.

In this sense Carleman, in his paper [15] in 1939, marked a true milestone, because he proved that the 2D elliptic operator $\mathcal{L}(u) = \Delta u + a(x)u$, where a is a bounded function, enjoys the UCP. At the same time Carleman conceived a highly constructive method that wide opened the doors to quantitative estimates of unique continuation for equations with nonanalytic coefficients. Since the 1950s the investigation on UCP has been extended to more general differential operators with a special attention to the regularity, in the first place, of the coefficients of the principal part of the operators. For instance, it was proved in [53], see also [49], [50], that the UCP for the second order elliptic equations doesn't hold true if the coefficients of principal part is in $C^{0,\alpha}(\mathbb{R}^n)$ for $\alpha < 1$ and $n > 2$. On the other side, the UCP applies when the coefficients of the principal part are Lipschitz continuous (see [7], [28]) and, consequently, a Hölder type propagation of smallness can be proved in the form of three-sphere inequality ([39]). We refer to [2] for an extensive and detailed analysis of connection between the UCP and propagation of smallness for second order elliptic equation. We should mention that the UCP for the second order elliptic equations with two variables with L^∞ coefficients can be deduced from the theory of quasiconformal mappings ([13], [56], [1]).

In the parabolic context, broadly speaking, the investigation about UCP is focused on two main topics: (i) backward uniqueness and backward stability estimates, (ii) spacelike unique continuation properties (which include the noncharacteristic Cauchy problem) and their quantitative versions. In this paper we concentrate on the second issue. For backward uniqueness and stability we refer to [36], [62], [64].

Let us consider the operator

$$\mathcal{L}(u) = \operatorname{div}(A(x,t)Du) - \partial_t u, \tag{1.2}$$

where $A(x,t)$, $(x,t) \in \mathbb{R}^{n+1}$, is a symmetric $n \times n$ matrix which we assume uniformly elliptic. The spacelike UCP has the following formulation: let $\widetilde{\Omega}$ be any open subset in Ω and let $J \subset (0,T)$ an interval or a single point; we say that $\mathcal{L}$ enjoys the spacelike UCP if

$$\mathcal{L}(u) = 0 \text{ in } \Omega \times (0,T) \text{ and } u = 0 \text{ on } \widetilde{\Omega} \times J \quad \text{imply} \quad u = 0 \text{ on } \Omega \times J. \tag{1.3}$$