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Non-integrability of the spacial n-center problem ☆

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Abstract

We prove the non-integrability of the spacial n-center problem. In order to prove it, we focus on the singularity of the differential equations extended to the complex space and then apply the Morales–Ramis theory to it. We also show the non-integrability of the spacial restricted $n+1$-body problem.

MSC: 37J30; 70F16

Keywords: Integrability; The n-center problem; Hamiltonian systems

1. Introduction

Let $H : \mathcal{D} \to \mathbb{R}$ be a smooth function where $\mathcal{D}$ is an open set in $\mathbb{R}^{2k}$. The Hamiltonian system is represented by the ordinary differential equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}(\boldsymbol{q}, \boldsymbol{p}), \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}(\boldsymbol{q}, \boldsymbol{p}) \qquad (j = 1, \dots, k) \tag{1}$$

where $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \dots, q_k, p_1, \dots, p_k) \in \mathcal{D}$. The function H is called the Hamiltonian and the natural number k is called the degrees of freedom.

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A function $F : \mathcal{D} \to \mathbb{R}$ is called the first integral of (1) if F is conserved along each solution of (1). The Poisson bracket of two functions $F, G : \mathcal{D} \to \mathbb{R}$ is the function defined by

$$\{F, G\} = \sum_{k=1}^{k} \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right).$$

A function $F : \mathcal{D} \to \mathbb{R}$ is a first integral of (1) if and only if $\{F, H\}$ is identically zero. Hamiltonian system (1) is called integrable if there are k first integrals $F_1(= H), F_2, \dots, F_k$ such that $dF_1, \dots, dF_k$ are linearly independent in an open dense set of $\mathcal{D}$ and that $\{F_i, F_j\}$ is identically zero for any $i, j = 1, \dots, k$.

The behavior of the orbits of integrable systems can be understood as quasi-periodic orbits on k-dimensional tori (see [1, Chapter 10]) while the dynamics of the non-integrable Hamiltonian systems are thought to be chaotic. Therefore it is an important subject to determine whether a given Hamiltonian is integrable or non-integrable.

This subject have been studied for centuries. Several approaches have been attempted for proving the integrability of some Hamiltonians. Noether theorem states that if a Hamiltonian has some symmetry, it has some first integrals. For example, from the fact that the central force systems have the rotating symmetry, the angular momentum is a first integral. As another method, if the Hamilton–Jacobi equation can be solved, the Hamiltonian with new coordinates can be represented as a function which depends only on the momentum variables. For example, the Hamilton–Jacobi equation of the two-center problem is separable, and hence can be solved. Therefore the two-center problem is integrable. As an example that the Hamilton–Jacobi equation is not separable, but that the Hamiltonian is integrable, Toda lattice is well known.

On the other hand, some methods for showing the non-integrability have been developed. Bruns [2] proved that in the 3-body problem there is no additional first integral which is represented by an algebraic function. After that, Poincaré [3] proved that for the perturbed Hamiltonian systems, there is no analytic first integral which also depends analytically on the parameter. Then by applying it to the restricted 3-body problem, he proved the non-existence of an analytic first integral depending analytically on a mass parameter.

Another theory in this field was originated by Kovalevskaya [4]. By focusing on singularities, she discovered new integrable parameters for the rigid body model. As a development of her approach, Ziglin [5,6] established the theory of the monodromy group for proving the non-integrability. By applying the Ziglin analysis, Yoshida [7] provided criteria for the non-integrability of the homogeneous Hamiltonian systems. Morales-Ruiz and Ramis [8,9] established a stronger theory by applying the differential Galois theory (Picard–Vessiot theory).

The first non-integrability proof of three-body problem using the Ziglin analysis was given in [10]. The proof with the differential Galois theory was given by [11]. Interesting results concerning some n-body problems using differential Galois integrability obstructions are also in [12]. Maciejewski and Przybylska [13] proved the non-integrability of the three-body problem for any fixed masses by using the Morales–Ramis theory. In order to prove it, they focused on the homothetic solutions and analyzed the variational equations along it.

In this paper, we show the non-integrability of the spatial n-center problem. Fix n positive constants m_k and n distinct points $\boldsymbol{c}_k \in \mathbb{R}^d$, and let

$$U(\boldsymbol{q}) = -\sum_{k=1}^{n} \frac{m_k}{|\boldsymbol{q} - \boldsymbol{c}_k|} = -\sum_{k=1}^{n} \frac{m_k}{\sqrt{(\boldsymbol{q} - \boldsymbol{c}_k) \cdot (\boldsymbol{q} - \boldsymbol{c}_k)}}.$$

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