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Spectral stability for the wave equation with periodic forcing

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Abstract

We consider the spectral stability problem for Floquet-type systems such as the wave equation $v_{\tau\tau} = \gamma^2 v_{xx} - \psi v$ with periodic forcing ψ. Our approach is based on a comparison with finite-dimensional approximations. Specific results are obtained for a system where the forcing is due to a coupling between the wave equation and a time-period solution of a nonlinear beam equation. We prove (spectral) stability for some period and instability for another. The finite-dimensional approximations are controlled via computer-assisted estimates.

1. Introduction

The aim of this paper is to develop a nonperturbative method for analyzing the stability of certain periodically driven systems. We do this in the context of a wave equation

$$v_{\tau\tau}(\tau, x) = \gamma^2 v_{xx}(\tau, x) - \psi(\tau, x)v(\tau, x), \qquad \tau \in \mathbb{R}, \qquad x \in (0, \pi), \tag{1.1}$$

where ψ depends periodically on the time variable τ. For the function v we impose Dirichlet boundary conditions at $x = 0$ and $x = \pi$. We consider a model where the coefficient ψ is deter-

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mined canonically by the desired time-period T. In this case we prove spectral stability for some value of T and absence of spectral stability for another.

We say that the equation (1.1) is spectrally stable if the corresponding evolution operator $\Phi(T)$ has no spectrum outside the unit circle. To be more specific, let us write the second order equation (1.1) in the usual way as a pair of first order (in τ) equations: $v_\tau = \nu$ and $\nu_\tau = \gamma^2 v_{xx} - \psi v$. The solution depends linearly on the initial condition at time zero, and this defines the time-τ map $\Phi(\tau)$ via the equation

$$V(\tau) = \Phi(\tau)V(0)\,, \qquad V(\tau) = \begin{bmatrix} v(\tau,.) \\ \nu(\tau,.) \end{bmatrix}, \qquad \tau \in \mathbb{R}\,. \tag{1.2}$$

If ψ is time-periodic with period T, then the flow Φ satisfies $\Phi(\tau + T) = \Phi(\tau)\Phi(T)$. So the growth properties of Φ are determined by the properties of the linear operator $\Phi(T)$. We note that, formally, this operator is symplectic, and thus its spectrum is invariant under complex conjugation $z \mapsto \bar{z}$ and inversion $z \mapsto 1/z$.

Our spectral analysis of Φ involves a comparison principle for monotone families of Floquet systems. This allows us e.g. to bound the eigenvalues of $\Phi(T)$ on the unit circle from both sides by the eigenvalues obtained from certain finite-dimensional approximations. The finite-dimensional systems are still nontrivial, but we can estimate their Floquet spectrum by using computer-assisted techniques.

Our analysis was motivated in part by numerical observations [6] on instabilities in a model of a suspension bridge. To be more precise, and to motivate our choice of the forcing ψ in (1.1), consider the following (Hamiltonian) system of partial differential equations:

$$\begin{aligned} u_{\tau\tau} &= -u_{xxxx} + \tfrac{1}{2}\big[f(u+v) + f(u-v)\big]\,, \\ v_{\tau\tau} &= \gamma^2 v_{xx} + \tfrac{1}{2}\big[f(u+v) - f(u-v)\big]\,. \end{aligned} \tag{1.3}$$

Here $u = u(\tau, x)$ and $v = v(\tau, x)$ are functions on $\mathbb{R} \times (0, \pi)$, satisfying Navier and Dirichlet boundary conditions, respectively, at $x = 0$ and $x = \pi$. The coupling function f is nonlinear and will be specified below.

The equations (1.3) are a simplified version of a model [6] for a suspension bridge. In this context, u describes the longitudinal modes of the bridge, and v describes the torsional modes. The function f models the force that the hangers apply to the deck; see also equation (4) and the ensuing discussion in [5]. Numerical studies on the model described in [6] indicate that there is a loss of stability in the torsional modes as the energy of the longitudinal modes exceeds a certain threshold. Since the torsional amplitudes are typically small, we will v-linearize the system (1.3) in the sense of dropping all terms of order v^2.

A reasonable choice for a simplified bridge model is $f(u) = -\kappa u - u^3$. With this choice of f, setting $v = 0$ in (1.3) reduces the system to a nonlinear beam equation for u. In order to show that this equation has a time-periodic solution with a given periods T, it is convenient to perform a change of variables $t = \alpha\tau$ with $\alpha = 2\pi/T$, so that T-periodicity in τ corresponds to 2π-periodicity in t. In these new variables, and for $f(u) = -\kappa u - u^3$, the system (1.3) becomes

$$\alpha^2 u_{tt} = -u_{xxxx} - \big(u^2 + \kappa\big)u\,, \tag{1.4}$$

$$\alpha^2 v_{tt} = \gamma^2 v_{xx} - \big(3u^2 + \kappa\big)v\,, \tag{1.5}$$