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ScienceDirect
J. Differential Equations 265 (2018) 2502–2543
Journal of Differential Equations
www.elsevier.com/locate/jde

Some unexpected results on the Brillouin singular equation: Fold bifurcation of periodic solutions

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Received 19 September 2017; revised 19 April 2018
Available online 8 May 2018

Abstract

In this paper, we find some new patterns regarding the periodic solvability of the Brillouin electron beam focusing equation

$$\ddot{x} + \beta(1 + \cos(t))x = \frac{1}{x}.$$

In particular, we prove that there exists $\beta^* \approx 0.248$ for which a 2π-periodic solution exists for every $\beta \in (0, \beta^*]$, and the bifurcation diagram with respect to β displays a *fold* for $\beta = \beta^*$. This result significantly contributes to the discussion about the well-known conjecture asserting that the Brillouin equation admits a periodic solution for every $\beta \in (0, 1/4)$, leading to doubt about its truthfulness. For the first time, moreover, we prove *multiplicity* of periodic solutions for a range of values of β near β^*. The technique used relies on rigorous computation and can be extended to some generalizations of the Brillouin equation, with right-hand side equal to $1/x^p$; we briefly discuss the cases $p = 2$ and $p = 3$.

MSC: 34B16; 34C25; 65G40; 37M20

Keywords: Brillouin focusing beam equation; Computer-assisted proof; Fold bifurcation; Periodic solutions; Non-autonomous singular ODEs

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https://doi.org/10.1016/j.jde.2018.04.042
0022-0396/

1. Introduction and main results

The Brillouin electron beam focusing equation

$$\ddot{x} + \beta(1 + \cos(t))x = \frac{1}{x}, \tag{1}$$

where β is a real parameter, has been object of study since some decades. Referring, e.g., to [1,2], it describes the dynamics of an electron beam generated by a shielded cathode (the electron beam being then a *Brillouin laminar flow*, whence the name of the equation) and guided by the magnetic field produced by a series of magnets arranged in a periodic way, with alternated polarity. A device of this kind, also called *electron gun*, finds application in many scientific, industrial and electronic machineries. We refer to [2, Chapter 6] and the references therein for further details about the physical motivations and possible generalizations of equation (1).

One of the main issues regarding equation (1) is to determine the values of β for which there exists a classical (thus one-signed) periodic solution. In view of the 2π-periodicity of the coefficients, the attention has focused since the very beginning on the existence of 2π-periodic (positive) solutions. There is a long dedicated literature (as we mention below, e.g., [3–9]), which has proceeded step-by-step in enlarging the solvability ranges in the parameter β. In particular, as far as we know, the best result of 2π-periodic solvability for equation (1) has been obtained in [5,6], and states that

$$\beta \in I_1 = (0, 0.1645028] \Longrightarrow \text{ equation (1) has a } 2\pi\text{-periodic solution.} \tag{2}$$

Of course, notice that for $\beta \leq 0$ no 2π-periodic solutions exist, as it immediately follows integrating the equation. The result (2) was achieved after a series of partial improvements (in chronological order, $\beta < 1/16$ [3], $\beta <\approx 0.1442$ [7], $\beta < 0.1532$ [8], $\beta <\approx 0.16448$ [9]), obtained exploiting the properties of the Green functions and the use of the Krasnoselski fixed-point theorem on cones.

However, the common conjecture which arose through the years [2], supported also by some numerical evidence (see Section 2), is that

$$\text{equation (1) possesses a } 2\pi\text{-periodic solution for every } \beta \in (0, 1/4)\ . \tag{3}$$

At the best of our knowledge, claim (3) is still unproved until now. Actually, the presence of the singularity, together with the difficulties in handling the linear term $\beta(1 + \cos t)x$, which at the same time may vanish or cross different resonance regions according to the time instant, make the usual (topological and variational) techniques not at all trivially applicable. Any progress in this direction thus seems to require a significant effort.

In the present paper, we address the 2π-periodic solvability of (1) with some recent techniques coming from rigorous computation. In particular, we display an unexpected feature of the solutions curve in a neighborhood of $\beta = 0.25$ as a possible explanation of the difficulties encountered in searching, even numerically, for a 2π-periodic solution therein: following the branch of 2π-periodic solutions as parametrized by β, *a fold bifurcation occurs close to* $\beta = 1/4$. At the same time, we are able to give a proof of the existence for values of β which are less than the "fold one", largely improving the range of existence from $(0, 0.1645028]$ to approximately $(0, 0.248]$. Precisely, we will prove the following result.