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On the regularity of the non-dynamic parabolic fractional obstacle problem

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Abstract

In the class of the non-dynamic Fractional Obstacle Problems of parabolic type, it is shown existence, uniqueness, apriori bounds of the solution and optimal regularity of the space derivatives of the solution. Furthermore, at free boundary points of positive parabolic density, it is proven that the time derivative of the solution is Hölder continuous. At such free boundary points, space–time regularity of the corresponding free boundary is obtained for any fraction $s \in (0, 1)$.

MSC: 35R45; 35R35; 49J40; 49K20; 91G80

Keywords: Parabolic obstacle problems; Hole-filling method; Free boundary regularity

1. Introduction

Considerable literature was dedicated to obstacle problems for the Lévy process i.e.

$$\min(\partial_t u + (-\Delta)^s, u - \psi) = 0 \quad \text{in } \mathbb{R}^{n-1}$$

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motivated by problems in optimal control and financial engineering. In this work our motivation is guided by the Signorini problem or the semi-permeable membrane model. In this case $\mathbb{R}^{n-1}$ is our physical boundary and represents a plane ($n = 3$ case) where the elastic set is lying or a chemical substance (or heat) is flowing across a semi-permeable membrane into Ω. The set Ω is composed of a non-homogeneous anisotropic material where its evolution is governed by

$$\partial_t u - \frac{1}{x_n^\gamma} \operatorname{div}(x_n^\gamma \nabla u) = 0 \quad \text{in } \Omega.$$

This induces a fractional heat equation

$$H^s u := (\partial_t - \Delta)^s u = 0 \quad \text{on } \mathbb{R}^{n-1}$$

where $\gamma = 1 - 2s$, for $0 < s < 1$. This is an example of a Master equation $Mu = 0$ (see [10]) which appears in the study of continuous time random walks where the random jumps occur with random time lag. We call our problem, as in the $\gamma = 0$ case (see [4]), non-dynamical in order to distinguish it from the ones where the processes are governed by diffusion equations of the type

$$\partial_t u + Mu = 0 \quad \text{on } \mathbb{R}^{n-1}$$

(see [18] and [19]).

In §2 a number of known properties of solutions to degenerate parabolic equations and to equations involving fractional powers of the heat operator are included which will be used in the subsequent sections. The formulation of our problem follows the generic scheme of parabolic obstacle problems and existence, uniqueness as well as basic properties are presented in §3. In §4 we prove Hölder continuity of space derivatives for any $0 < s < 1$; thus extending the result of $s = \frac{1}{2}$ case (see [1]). Then in §5 via a monotonicity formula similar to the one in [4] we obtain the optimal regularity of space derivatives. Since, in general, the time derivative can experience jumps at free boundary points, for instance when the solution hits the obstacle for the first time, we prove that at points of positive parabolic density with respect to the coincidence set the time derivative is continuous nearby (§6). The quasi-convexity introduced in [4] together with the continuity of the time derivative of §6 allows us to use the elliptic free boundary theory developed in [5] and [9], using the appropriate elliptic Almgren's frequency formula. Therefore we are able to obtain (§7) space–time smoothness of the free boundary at regular points for any $0 < s < 1$. This is feasible since, in our case, the diffusion term is always dominant, as opposed to the one for the Lévy processes mentioned above where the free boundary theory breaks down at $s = \frac{1}{2}$.

2. Preliminaries

We start this section by presenting certain known properties of solutions to degenerate parabolic equations which will be of use in the sequel. We consider degenerate parabolic operators of the form

$$\operatorname{div}(w(x', y) A \nabla u(x', y, t)) - w(x', y)\partial_t u(x', y, t) = 0 \tag{2.1}$$

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