Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence of solution to parabolic equations with mixed boundary condition on non-cylindrical domains

Tujin Kim [a,*], Daomin Cao [b,c]

[a] *Institute of Mathematics, Academy of Sciences, Pyongyang, Democratic People's Republic of Korea*
[b] *Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, PR China*
[c] *University of Chinese Academy of Sciences, Beijing 100049, PR China*

Received 23 June 2014; revised 24 October 2017

Abstract

In this paper we study the existence of weak solutions to initial boundary value problems of linear and semi-linear parabolic equations with mixed boundary conditions on non-cylindrical domains $\bigcup_{t\in(0,T)}\Omega(t)\times\{t\}$ of spatial-temporal space $\mathbb{R}^N\times\mathbb{R}$. In the case of the linear equation, each boundary condition is given on any open subset of the boundary surface $\Sigma=\bigcup_{t\in(0,T)}\partial\Omega(t)\times\{t\}$ under a condition that the boundary portion for Dirichlet condition $\Sigma_0\subset\Sigma$ is nonempty at any time t. Due to this, it is difficult to reduce the problem to the one on a cylindrical domain by diffeomorphism of the spatial domains $\Omega(t)$. By a transformation of the unknown function and the penalty method, we connect the problem to a monotone operator equation for functions defined on the non-cylindrical domain. We are also concerned with a semilinear problem when the boundary portion for Dirichlet condition is cylindrical.

MSC: 35K20; 35K55; 35A15

Keywords: Parabolic equation; Non-cylindrical domain; Mixed boundary condition; Existence

* Corresponding author.
E-mail addresses: math.tujin@star-co.net.kp (T. Kim), dmcao@amt.ac.cn (D. Cao).

https://doi.org/10.1016/j.jde.2018.04.046
0022-0396/

1. Introduction

Let $\Omega(t)$ be bounded connected domains of $\mathbb{R}^N$ and $Q = \bigcup_{t\in(0,T)} \Omega(t) \times \{t\},\ 0 < T < \infty$. We are concerned with the initial boundary value problems of linear and semi-linear parabolic equations on the non-cylindrical domains $\bigcup_{t\in(0,T)} \Omega(t) \times \{t\}$ with mixed boundary conditions on the boundary surfaces $\Sigma = \bigcup_{t\in(0,T)} \partial\Omega(t) \times \{t\}$. In the special case such that $Q = \Omega \times (0, T)$, portions of $\partial\Omega \times (0, T)$ for boundary conditions may be non-cylindrical surfaces which means that the boundary portions are not expressed by $\Gamma \times (0, T)$ with some $\Gamma \subset \partial\Omega$.

There are a lot of papers devoted to the study of parabolic differential equations on non-cylindrical domains of spatial-temporal space and various methods have been used for this. In [6] for a linear equation with homogeneous Dirichlet boundary condition the energy inequality is obtained, and the unique existence of a solution is proved. For the linear (cf. [10], [15] and [16]), semilinear (cf. [14]) and nonlinear (cf. [15]) equations on domains expanding in time, the existence and uniqueness of solutions to the initial boundary value problems with homogeneous Dirichlet boundary condition are studied. For such a domain and the boundary condition [14] also deals with an attractor of a semilinear heat equation and [2] shows the unique existence of a solution to a linear Schrödinger-type equation. Dealing with the Dirichlet condition, authors of [3] assume only Hölder continuity in time of the boundary. In [26] a semigroup-theoretical result is obtained and the result is applied to the initial boundary value problem of a linear parabolic equation with inhomogeneous Dirichlet boundary condition on a non-cylindrical domain. Applying the potential method, some authors study the unique existence of solutions to the initial boundary value problems of linear equations (see [8] and references therein). Domains in [21] and [22], where existence, uniqueness and regularity are studied, are more general, that is, "initial" condition is given on a hypersurface of spatial-temporal space instead of the plane $t = 0$. In [18] the optimal regularity of solution to a special kind of 1-dimensional problem is considered. Neumann problem of the heat equations (cf. [12]) and Robin problem of a parabolic equation (cf. [13]) on the non-cylindrical domains are studied. In [24] and [27] an optimal control and controllability of parabolic equations with homogenous Dirichlet condition on non-cylindrical domains, respectively, are studied.

Also, there are many papers for the initial boundary value problems with mixed boundary conditions.

The initial boundary value problems of linear parabolic equations with mixed time dependent lateral boundary conditions on cylindrical domains are studied (cf. [4], [5], [29]). The boundary conditions on the lateral surfaces in [4] may be two among Dirichlet, Neumann and Robin conditions, but the boundary portion for a kind of boundary condition is a connected and relatively open subset of the lateral surface and the boundary of the portion is tangent to the plane $t = 0$. The first part of [5] is concerned with a classical problem on cylindrical domains and the result is applied to the problem with zero initial condition and lateral mixed boundary conditions. In [5] the lateral boundary surface is also divided into two non-cylindrical portions and one portion is a relatively open connected subset of the boundary surface and at each point of the portion it is transverse to the plane $t = const$, and the non-cylindrical portions are transformed to cylindrical ones by a diffeomorphism. In [29] first the linear parabolic problems on cylindrical domains with mixed variable inhomogeneous Dirichlet and Neumann conditions are studied. Here the change in time of distances of the sections at times t of the boundary portion for Dirichlet condition must be dominated by a Lipschitz continuous function in time t. In [29] then as an application of the result the unique existence of solution to a linear parabolic problem with homogeneous Dirichlet boundary condition on non-cylindrical domains is obtained.

Please cite this article in press as: T. Kim, D. Cao, Existence of solution to parabolic equations with mixed boundary condition on non-cylindrical domains, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.04.046