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Extremal domains on Hadamard manifolds

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Abstract

We investigate the geometry and topology of extremal domains in a Hadamard manifold, i.e., domains that support a positive solution to an overdetermined elliptic problem (OEP).

First, we study narrow properties of such domains and characterize the boundary at infinity. We give an upper bound for the Hausdorff dimension of its boundary at infinity and how the domain behaves at infinity. This shows interesting relations with the Singular Yamabe Problem.

Later, we focus on extremal domains in the Hyperbolic Space. Symmetry and boundedness properties will be shown. In a certain sense, we extend Levitt–Rosenberg's Theorem to OEPs, which suggests a strong relation with constant mean curvature hypersurfaces. In particular, we are able to prove the Berestycki–Caffarelli–Nirenberg Conjecture under certain assumptions either on the boundary at infinity of the extremal domain or on the OEP itself.

Also a height estimate for solutions on extremal domains in a Hyperbolic Space will be given.

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Keywords: The moving plane method; Overdetermined problems; Maximum principle; Neumann conditions; Hyperbolic space

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1. Introduction

Alexandrov [1] introduced *the moving plane method* and used it to prove a very classical result in the theory of constant mean curvature (CMC for short) hypersurfaces: *the only compact CMC hypersurfaces embedded in the Euclidean n-space $\mathbb{R}^n$ are spheres*. By also applying the moving plane method and meanwhile improving the boundary point maximum principle to a more delicate version (cf. [42, Lemma 1]), Serrin [42] proved that if the OEP

$$\begin{cases} \Delta u = -1 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \langle \nabla u, \vec{v} \rangle_{\mathbb{R}^n} = \alpha & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then Ω must be a ball, where Ω is a bounded open connected domain in $\mathbb{R}^n$, $\vec{v}$ the unit outward normal vector of the boundary $\partial\Omega$, $\langle \cdot \cdot \rangle_{\mathbb{R}^n}$ the inner product in $\mathbb{R}^n$, and α a non-positive constant. Serrin's result is of great importance, since it made the moving plane method available to a large part of the mathematical community. If the constant -1 in the first equation of the above OEP (1.1) is replaced by a function f with Lipschitz regularity, Pucci and Serrin [35] can also get the symmetry result, i.e., the domain Ω must be a ball in $\mathbb{R}^n$. The OEP has wide applications in physics, which can be used to describe physical phenomenons. For instance, if the constant -1 in (1.1) is replaced by some constant $-k$ depending on the viscosity and the density of a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω, and moreover, if we set up rectangular coordinates (x, y, z) with the z-axis directed along the pipe, then the velocity u of this flow satisfies the equation

$$\Delta u + k = 0$$

with the boundary condition $u = 0$ on $\partial\Omega$. Applying Serrin's result, we can claim that the tangential stress per unit area on the pipe wall, which is represented by $\mu \langle \nabla u, \vec{v} \rangle_{\mathbb{R}^n}$ where μ is the viscosity, is the same at all points of the wall if and only if it has a circular cross section. Besides, in the linear theory of torsion of a solid straight bar of cross section Ω, and also in the Signorini problem, the OEP introduced above is related to the physical models therein (see, e.g., [21,44] for the details).

We know that if one imposes suitable conditions on the separation interface of the variational structure, overdetermined boundary conditions naturally appear in free boundary problems (see, for instance, [2]). In this process, several methods based on blow-up techniques applied to the intersection of Ω with a small ball centered at a point of $\partial\Omega$ were used to locally study the regularity of solutions of free boundary problems. This leads to the study of an elliptic equation in an unbounded domain. In this situation, Berestycki, Caffarelli and Nirenberg [4] considered the following OEP

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \langle \nabla u, \vec{v} \rangle_{\mathbb{R}^n} = \alpha & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$