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Degenerate SDEs with singular drift and applications to Heisenberg groups ☆

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Abstract

By using the ultracontractivity of a reference diffusion semigroup, Krylov's estimate is established for a class of degenerate SDEs with singular drifts, which leads to existence and pathwise uniqueness by means of Zvonkin's transformation. The main result is applied to singular SDEs on generalized Heisenberg groups.

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Keywords: Degenerate SDEs; Krylov estimate; Zvonkin transformation; Heisenberg group

1. Introduction

Since 1974 when Zvonkin [29] proved the well-posedness of the Brownian motion with bounded drifts, his argument (known as Zvonkin's transformation) has been developed for more general models with singular drifts, see [19,13,26,24] and references within for non-degenerate SDEs, and [4–7] and [11,20] for non-degenerate semilinear SPDEs. In these references only Gaussian noise is considered, see also [17,25] for extensions to the case with jump.

In recent years, Zvonkin's transformation has been applied in [2,16,22,23,27] to a class of degenerate SDEs/SPDEs with singular drifts. This type degenerate stochastic systems are called

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stochastic Hamiltonian systems in probability theory. Consider, for instance, the following SDE for (X_t, Y_t) on $\mathbb{R}^{2d}$ $(d \geqslant 1)$:

$$\begin{cases} \mathrm{d}X_t = Y_t \mathrm{d}t, \\ \mathrm{d}Y_t = b_t(X_t, Y_t)\mathrm{d}t + \sigma_t(X_t, Y_t)\mathrm{d}W_t, \end{cases} \tag{1.1}$$

where W_t is the d-dimensional Brownian motion, and

$$b : [0, \infty) \times \mathbb{R}^{2d} \to \mathbb{R}^d, \quad \sigma : [0, \infty) \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$$

are measurable. According to [27, Theorem 1.1], if there exists a constant $K > 1$ such that

$$K^{-1}|v| \leqslant |\sigma_t v| \leqslant K|v|, \quad t \geqslant 0, v \in \mathbb{R}^d,$$

and for some constant $p > 2(1 + 2d)$,

$$\sup_{t \geqslant 0} \|\nabla \sigma_t\|_{L^p(\mathbb{R}^{2d})} + \int_0^\infty \|(1 - \Delta_x)^{\frac{1}{3}} b_t\|^p_{L^p(\mathbb{R}^{2d})} \mathrm{d}t < \infty,$$

then the SDE (1.1) has a unique strong solution for any initial points. By a standard truncation argument, the existence and pathwise uniqueness up to the life time hold under the corresponding local conditions.

In this paper, we aim to extend this result to general degenerate SDEs, in particular, for singular diffusions on generalized Heisenberg groups. As typical models of hypoelliptic systems, smooth SDEs on Heisenberg groups have been intensively investigated, see for instance [1,8–10, 14,21] and references within for the study of functional inequalities, gradient estimates, Harnack inequalities, and Riesz transforms. We will use these results to establish Krylov's estimates for singular SDEs and to prove the existence and uniqueness of strong solutions using Zvonkin's transformation.

In Section 2, we present a general result (see Theorem 2.1(3)) for the existence and uniqueness of degenerate SDEs with singular drifts, and then apply this result in Section 3 to singular diffusions on generalized Heisenberg groups.

2. General results

For fixed constant $T > 0$, consider the following SDE on $\mathbb{R}^N$:

$$\mathrm{d}X_t = Z_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad t \in [0, T], \tag{2.1}$$

where B_t is the m-dimensional Brownian motion with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and

$$Z : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N, \quad \sigma : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N \otimes \mathbb{R}^m$$

are measurable and locally bounded. We are in particular interested in the case that $m < N$ such that this SDE is degenerate.