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On the decay rate for the wave equation with viscoelastic boundary damping

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Abstract

We consider the wave equation with a boundary condition of memory type. Under natural conditions on the acoustic impedance of the boundary a corresponding semigroup of contractions is known to exist. With the help of quantified Tauberian theorems we establish energy decay rates via resolvent estimates on the generator of the semigroup. Using a variational approach, we reduce resolvent estimates to estimates for a sesquilinear form induced by an operator characteristic function arising form the matrix representation of the generator. Under not too strict additional assumptions on the acoustic impedance we establish an upper bound on the resolvent. For the wave equation on the interval or the disc we prove our estimates to be sharp.

MSC: primary 35B40, 35L05, 47D06; secondary 35P20

Keywords: Wave equation; Viscoelastic damping; Memory; Decay rates; Resolvent estimates; C_0-semigroups

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $k : \mathbb{R} \to [0, \infty)$ be an nonzero integrable function, depending on the time-variable only and vanishing on $(-\infty, 0)$. We consider a model for the evolution of sound in a compressible fluid with viscoelastic surface [20]:

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$$\left.\begin{aligned} p_t(t,x) + \operatorname{div} v(t,x) &= 0 \quad (t \in \mathbb{R}, x \in \Omega), \\ v_t(t,x) + \nabla p(t,x) &= 0 \quad (t \in \mathbb{R}, x \in \Omega), \\ k * p(t,x) - n \cdot v(t,x) &= 0 \quad (t \in \mathbb{R}, x \in \partial\Omega). \end{aligned}\right\} \tag{1}$$

The convolution is defined by $k * p(t,x) = \int_{-\infty}^{t} k(t-s)p(s,x)\mathrm{d}s$. Here n is the outward normal vector of $\partial\Omega$, which exists almost everywhere for Lipschitz domains. The acoustic pressure and the fluid velocity are given by p and v respectively. The Laplace transform $\hat{k}$ of the kernel k is called the *acoustic impedance*. We assume that $k \in L^1(0,\infty)$ is a *completely monotonic function*. That is, there exists a positive Radon measure ν on $[0,\infty)$ such that k is the Laplace transform of that measure $k(t) = \int_{[0,\infty)} e^{-\tau t}\mathrm{d}\nu(\tau)$, $t > 0$. The integrability assumption on k is equivalent to (see e.g. [13, Theorem 2.5])

$$\nu(\{0\}) = 0 \text{ and } \int_0^\infty \tau^{-1}\mathrm{d}\nu(\tau) < \infty. \tag{2}$$

Desch et al. [11] reformulated (1) in the language of C_0-semigroups as an abstract Cauchy Problem $\mathbf{x}(t_0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(t) + \mathcal{A}\mathbf{x}(t) = 0$, $(t > t_0)$ on a Hilbert space $\mathcal{H}$. We sketch their approach in Section 2, where we assume without loss of generality that $t_0 = 0$. Here $\mathbf{x} = (p, v, \psi)$ and $\mathcal{A}$ is a three by three operator matrix. The function $\psi : \mathbb{R} \times [0,\infty) \times \partial\Omega \to \mathbb{C}$ is given by

$$\psi(t,\tau,x) = \int_{-\infty}^{t} e^{-\tau(t-s)} p(s,x)\mathrm{d}s. \tag{3}$$

We see that ψ incorporates memory effects at the boundary since it depends on the acoustic pressure at earlier times than t. The energy of the system is given by

$$E(\mathbf{x}(t)) = \int_\Omega |p(t,x)|^2 + |v(t,x)|^2 \,\mathrm{d}x + \int_0^\infty \int_{\partial\Omega} |\psi(t,\tau,x)|^2 \,\mathrm{d}S(x)\mathrm{d}\nu(\tau).$$

We also write $E(t,\mathbf{x}) = E(\mathbf{x}(t))$ and $E(\mathbf{x}_0) = E(\mathbf{x}(t_0))$. Below we will see that $E(\mathbf{x}_0) = \|\mathbf{x}_0\|_{\mathcal{H}}^2$ by definition of the state space $\mathcal{H}$, which is actually the motivation to define the energy like that.

Desch et al. showed that the operator $-\mathcal{A}$ is the generator of a C_0-semigroup of contractions. Moreover they proved that $\mathcal{A}$ is injective and that $\mathrm{i}\mathbb{R}\backslash\{0\}$ contains no spectrum. Thus, using the ABLV-theorem (see e.g. [4, Theorem 5.5.5]), they could deduce that the semigroup is strongly stable. This means that

$$E(t,\mathbf{x}_0) \to 0 \text{ as } t \to \infty$$

for every initial state $\mathbf{x}_0 \in \mathcal{H}$. It is natural to ask whether the energy decays *uniformly* (and thus necessarily exponentially by the semigroup law) with respect to the norm of the initial state. In a companion paper [12] Desch et al. could show that this is not true in general. In that paper they studied, in a slightly more general setting, the spectrum of $\mathcal{A}$ in three geometrical configurations: a line, a disc and a cylinder. By using a Rouché argument, in all cases they proved the existence

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