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Regularization around a generic codimension one fold-fold singularity

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Abstract

This paper is devoted to study the generic fold-fold singularity of Filippov systems on the plane, its unfoldings and its Sotomayor–Teixeira regularization. We work with general Filippov systems and provide the bifurcation diagrams of the fold-fold singularity and their unfoldings, proving that, under some generic conditions, is a codimension one embedded submanifold of the set of all Filippov systems. The regularization of this singularity is studied and its bifurcation diagram is shown. In the visible–invisible case, the use of geometric singular perturbation theory has been useful to give the complete diagram of the unfolding, specially the appearance and disappearance of periodic orbits that are not present in the Filippov vector field. In the case of a linear regularization, we prove that the regularized system is equivalent to a general slow-fast system studied by Krupa and Szmolyan [10].

Keywords: Non-smooth systems; Regularization; Bifurcations; Melnikov method; Singular perturbation theory

1. Introduction

In this paper, derived from the thesis [12], we study the generic fold-fold singularity of Filippov systems on the plane, its unfoldings and its regularization, more concretely, its Sotomayor–Teixeira regularization [15].

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The first part of the paper is devoted to study the fold-fold singularity. This singularity has been studied in [8] and [4] by considering some simple normal forms for the Filippov vector fields and their unfoldings, and also in the original book of Filippov [3]. A systematic study of the set of structurally stable Filippov vector fields was done in [4] but, besides the previously mentioned works, which study normal forms, there does not exist a rigorous approach to the codimension one singularities. Our goal, realized in Theorem 2.6, is to work with general Filippov systems and provide the bifurcation diagrams of these singularities and their unfoldings, proving that the set of the fold-fold singularities, under some generic conditions, is a codimension one embedded submanifold of Ξ_0, the set of structurally stable Filippov systems.

The second part of the paper is dedicated to study the regularization of the unfoldings of the fold-fold singularity and is a natural continuation of the paper [14], where Filippov vector fields near a fold-regular point were considered. It is known [1,18] that, under general conditions, in the so-called sliding and escaping regions, the regularized system has a normally hyperbolic invariant manifold, attracting near the sliding region and repelling near the escaping one. Furthermore, the flow of the regularized vector field reduced to this invariant manifold approaches to the Filippov flow. In [14], these results were extended to visible tangency points, using asymptotic methods following [13]. The work [6] extended these results to $\mathbb{R}^3$ by use of blow-up methods.

The results in this work are mostly given in [12], therefore the cumbersome computations are referred to it.

During the period of time of writing this paper, the work [7], where the authors study this problem, came out. In [7] the authors perform some changes of variables to simplify the system and then study the normal form obtained using blow-up methods and analyzing it in different charts (variables). Their analytic approach completely characterizes the existence and the attracting/repelling character of the equilibrium points showing that in some relevant cases, there is a curve in the parameter plane where the equilibrium of the system has a Hopf bifurcation. They also show that the (sub/super critical) character of the Hopf bifurcation depends on the considered Filippov vector fields but also on the regularization function. In fact, in formula (7.15) of that paper, the authors give an explicit formula for the Lyapunov coefficient at the Hopf bifurcation for the normal form system. They also study the appearance and character of the family of periodic orbits at the Hopf bifurcation and their evolution. In the invisible–invisible case, they succeed in describing the family as a smooth family of locally unique periodic orbits, that, in some cases, can undergo a saddle-node bifurcation. In the visible–invisible case, they prove the existence of a curve in the parameter plane where a Maximal Canard occurs. Moreover, they prove the existence of a family of locally unique "big" periodic orbits for parameters (exponentially) close to the Canard curve. The authors conjecture that the "small" curves near the Hopf bifurcation and the "big" curves near the Canard curve belong to the same smooth family of locally unique periodic orbits.

The approach in our paper is mainly topological providing some new and slightly different results which complement the ones obtained in [7]; one major goal is to give results directly checkable in a given system, for this reason we work in the original variables of the system, and we present its possible phase portraits. We use topological methods to get the generic conditions which determine the phase portrait in terms of some intrinsic and explicit quantities that can be computed directly from the original system. For this reason, although [7] already computed the values of the Hopf and Canard curves, we can not rely in their formulas (7.14) and (6.22) because they are only valid for systems in normal form and we have done these computations for general vector fields in Propositions 4.6, 3.7.

We now present these different results and the new ones presented in this paper.