ARTICLE IN PRESS YJDEQ:9289

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J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

Local minimizers over the Nehari manifold for a class of concave-convex problems with sign changing nonlinearity

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Received 22 June 2017; revised 9 April 2018

Abstract

We study a p-Laplacian equation involving a parameter λ and a concave-convex nonlinearity containing a weight which can change sign. By using the Nehari manifold and the fibering method, we show the existence of two positive solutions on some interval $(0, \lambda^* + \varepsilon)$, where λ^* can be characterized variationally. We also study the asymptotic behavior of solutions when $\lambda \downarrow 0$.

MSC: 35J65; 35B32; 35J50; 35J92

Keywords: Concave-convex; p-Laplacian; Variational methods; Bifurcation; Nehari manifold; Fibering method

1. Introduction

Consider the following equation

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + f|u|^{\gamma-2}u \text{in } \Omega, \\ \qquad u \in W_0^{1,p}(\Omega), \end{cases} \tag{P_λ}$$

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https://doi.org/10.1016/j.jde.2018.04.018
0022-0396/

Please cite this article in press as: K. Silva, A. Macedo, Local minimizers over the Nehari manifold for a class of concave-convex problems with sign changing nonlinearity, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.04.018

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1 boundary, $\lambda > 0$, $1 < q < p < \gamma < p^*$ and p^* is the critical Sobolev exponent, $f \in L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$ is the standard Sobolev space. We say that $u \in W_0^{1,p}(\Omega)$ is a solution of (P_λ) if u is a critical point for $\Phi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ where

$$\Phi_\lambda(u) := \frac{1}{p}\int |\nabla u|^p - \frac{\lambda}{q}\int |u|^q - \frac{1}{\gamma}\int f|u|^\gamma.$$

We denote $\|u\| = \left(\int |\nabla u|^p\right)^{1/p}$ as the standard Sobolev norm in $W_0^{1,p}(\Omega)$ and consider the following extremal value

$$\lambda^* \equiv \frac{\gamma - p}{\gamma - q}\left(\frac{p-q}{\gamma-q}\right)^{\frac{p-q}{\gamma-p}} \inf_{u \in W_0^{1,p}\setminus\{0\}} \left\{ \frac{\|u\|^{p\frac{\gamma-q}{\gamma-p}}}{\|u\|_q^q F(u)^{\frac{p-q}{\gamma-p}}} : F(u) > 0 \right\},$$

where $F(u) = \int f|u|^\gamma$. Let $z \in W_0^{1,p}(\Omega)$ be the unique positive solution of the Lane–Emden equation

$$\begin{cases} -\Delta_p u = |u|^{q-2}u & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The main result of this work is the following

Theorem 1.1. *Assume that $f^+ := \max\{f(x), 0\} \not\equiv 0$. There exists $\varepsilon > 0$ such that for all $\lambda \in (0, \lambda^* + \varepsilon)$ the problem (P_λ) has two positive solutions w_λ, u_λ. Moreover*

(i) $D_{uu}\Phi_\lambda(w_\lambda)(w_\lambda, w_\lambda) < 0$, $D_{uu}\Phi_\lambda(u_\lambda)(u_\lambda, u_\lambda) > 0$;
(ii)
$$\lim_{\lambda \downarrow 0} \frac{u_\lambda}{\lambda^{\frac{1}{p-q}}} = z.$$

When $p = 2$ and $f \equiv 1$ the problem (P_λ) was studied by Ambrosetti–Brezis–Cerami in [1]. There, among other things, they show the existence of $\Lambda > 0$ such that for all $\lambda \in (0, \Lambda)$ the problem (P_λ) has at least two positive solutions, while for $\lambda = \Lambda$ it has at least one positive solution and for $\lambda > \Lambda$ there is no positive solution for (P_λ). To find the first solution they used the sub and super solution method, while for the second solution they used the mountain pass theorem. Moreover, from the sub and super solution method, one can easily see that the first branch of solutions which bifurcates from 0 satisfies property (**ii**). Later on, there was some improvement in Ambrosetti–Azorero–Peral [2], where the authors proved the existence of some Λ satisfying the above properties, however for $p > 1$, $f \equiv 1$ and Ω a ball. Finally, the result was generalized for $p > 1$ by Azorero–Peral–Manfredi in [3].

More recently, some authors studied the problem (P_λ) by using only variational methods, to wit, the Nehari manifold (see Nehari [4,5]) and the fibering method of Pohozaev [6]. Among these authors we can cite the work of Il'yasov [7], which considered the problem (P_λ) with $0 \le f \in L^d(\Omega)$ and $p > 1$. He was able to show the existence of a parameter $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$ the problem (P_λ) has two positive solutions. In [8] Brown–Wu considered the case $p = 2$ and a indefinite nonlinearity, that is, f change sign in Ω. By minimizing over the Nehari manifold they proved the existence of two positive solutions for small λ.

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