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# Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes

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## Abstract

In this paper, we are concerned with the fractional order static Hartree equations with critical nonlocal nonlinearity. We prove that the positive solutions are radially symmetric about some point in $\mathbb{R}^d$ and must assume the certain explicit forms. The arguments used in our proof is a variant (for nonlocal nonlinearity) of the direct moving plane method for fractional Laplacians in [6]. The main ingredients are the variants (for nonlocal nonlinearity) of the maximum principles, i.e., *Decay at infinity* and *Narrow region principle* (Theorem 2.1 and 2.6).


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## 1. Introduction

In this paper, we consider the following critical fractional order static Hartree equations with nonlocal nonlinearity

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u=\Big(\frac{1}{|x|^{2\alpha}}*|u|^2\Big)u, \quad x\in\mathbb{R}^d,\\ u\in C^{1,1}_{loc}\cap\mathcal{L}_\alpha(\mathbb{R}^d), \quad u(x)>0, \quad x\in\mathbb{R}^d,\end{cases} \tag{1.1}$$

where $0<\alpha<\min\{2,\frac{d}{2}\}$, $d\geq 2$ and

$$\mathcal{L}_\alpha(\mathbb{R}^d):=\Big\{u:\mathbb{R}^d\to\mathbb{R}\,\Big|\int_{\mathbb{R}^d}\frac{|u(x)|}{1+|x|^{d+\alpha}}dx<\infty\Big\}.$$

The nonlocal fractional Laplacians are defined by (see [2,6,9,31,36])

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=C_{\alpha,d}\,P.V.\int_{\mathbb{R}^d}\frac{u(x)-u(y)}{|x-y|^{d+\alpha}}dy:=C_{\alpha,d}\lim_{\epsilon\to 0}\int_{|y-x|\geq\epsilon}\frac{u(x)-u(y)}{|x-y|^{d+\alpha}}dy \tag{1.2}$$

for functions $u\in C^{1,1}_{loc}\cap\mathcal{L}_\alpha(\mathbb{R}^d)$, where the constant $C_{\alpha,d}=\big(\int_{\mathbb{R}^d}\frac{1-\cos(2\pi\zeta_1)}{|\zeta|^{d+\alpha}}d\zeta\big)^{-1}$.

One should observe that both the fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$ and the Hartree type nonlinearity are nonlocal in our equation (1.1), which is closely related to the following integral equation

$$u(x)=\int_{\mathbb{R}^d}\frac{R_{\alpha,d}}{|x-y|^{d-\alpha}}\Big(\int_{\mathbb{R}^d}\frac{|u(z)|^2}{|y-z|^{2\alpha}}dz\Big)u(y)dy, \tag{1.3}$$

where the Riesz potential's constants $R_{\alpha,d}:=\frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac{d}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}$ (see [32]). We say that equations (1.1) and (1.3) are $\dot{H}^{\frac{\alpha}{2}}$-critical in the sense that both the equations (1.1) and (1.3) and the $\dot{H}^{\frac{\alpha}{2}}$ norm are invariant under the same scaling $u_\rho(x)=\rho^{\frac{d-\alpha}{2}}u(\rho x)$.

PDEs of type (1.1) arise in the Hartree–Fock theory of the nonlinear Schrödinger equations (see [27]). The solution $u$ to problem (1.1) is also a ground state or a stationary solution to the following $\dot{H}^{\frac{\alpha}{2}}$-critical focusing fractional order dynamic Schrödinger–Hartree equation

$$i\partial_t u+(-\Delta)^{\frac{\alpha}{2}}u=\Big(\frac{1}{|x|^{2\alpha}}*|u|^2\Big)u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d. \tag{1.4}$$

The Schrödinger–Hartree equations have many interesting applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules (see, e.g. [16]). Dynamic equations of the type (1.4) have been quite intensively studied, please refer to [25,29] and the references therein, in which the ground state solution can be regarded as a crucial criterion or threshold for global well-posedness and scattering in the focusing case. Therefore, the classification of solutions to (1.1) plays an important and fundamental role in the study of the focusing fractional order Schrödinger–Hartree equations (1.4).