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J. Differential Equations 265 (2018) 2191–2242

Journal of Differential Equations

www.elsevier.com/locate/jde

Transition fronts of time periodic bistable reaction–diffusion equations in $\mathbb{R}^N$

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Received 17 December 2016; revised 25 December 2017
Available online 17 April 2018

Abstract

This paper is concerned with the existence and qualitative properties of transition fronts for time periodic bistable reaction–diffusion equations in $\mathbb{R}^N$. We first show that any almost-planar transition front is actually planar, regardless of the number of transition layers. Then we prove that all transition fronts admit a global mean speed γ and it holds $\gamma = |c|$, where c is the speed of the planar traveling front. Finally we establish the existence of a transition front in $\mathbb{R}^N$ that is not a standard traveling front. Such a front behaves like three moving time periodic planar fronts as time goes to $-\infty$ and like a time periodic V-shaped traveling front as time goes to ∞.

MSC: 35C07; 35K57; 35B08

Keywords: Bistable; Transition fronts; Time periodic; Reaction–diffusion equations

1. Introduction

This paper is concerned with the existence and qualitative properties of transition fronts to the following reaction–diffusion equation:

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https://doi.org/10.1016/j.jde.2018.04.032
0022-0396/

$$u_t = \Delta u + f(t,u), \quad x \in \mathbb{R}^N, \; t \in \mathbb{R}. \tag{1.1}$$

In the sequel, we assume that the following hypotheses hold.

(H1) $f(t,u) \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ is T-periodic in t, i.e., there exists a constant $T > 0$ such that $f(t + T, u) = f(t,u)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}$.

(H2) The period map $P(\alpha) := \omega(T,\alpha)$ has exactly three fixed points $0, \alpha^0, 1$ such that $0 < \alpha^0 < 1$, where $\omega(t,\alpha)$ is the solution of

$$\omega_t = f(t,\omega), \; t \in \mathbb{R}, \; \omega(0,\alpha) = \alpha \in \mathbb{R}.$$

Furthermore, they are non-degenerate and 0 and 1 are stable, i.e.,

$$\frac{d}{d\alpha}P(0) < 1, \quad \frac{d}{d\alpha}P(1) < 1 \; \text{ and } \; \frac{d}{d\alpha}P(\alpha^0) > 1.$$

A typical example satisfying the hypotheses (H1)–(H2) is the nonlinearity $f = u(u - a(t))(1-u)$ with $a(t) = a(t+T)$ and $0 < a(t) < 1$. More general example can be referred to Alikakos et al. [1].

Throughout this paper, the solution $u : \mathbb{R} \times \mathbb{R}^N \to [0,1]$ is understood as a classical solution of (1.1). As a result of the strong maximum principle, there should be either $u \equiv 0$ or $0 < u < 1$ or $u \equiv 1$. We focus on the nontrivial case in the sequel, namely the solution $0 < u < 1$.

Under the assumptions (H1)–(H2), Alikakos et al. [1] showed that (1.1) admits a *time periodic planar traveling front* $\phi(t, x \cdot e - ct)$ satisfying

$$\begin{cases} \phi_t - c\phi_\xi - \phi_{\xi\xi} - f(t,\phi) = 0, \; (t,\xi) \in \mathbb{R}^2, \\ \phi(t,\infty) = \lim_{\xi \to \infty} \phi(t,\xi) = 0, \; t \in \mathbb{R}, \\ \phi(t,-\infty) = \lim_{\xi \to -\infty} \phi(t,\xi) = 1, \; t \in \mathbb{R}, \\ \phi(t+T,\cdot) = \phi(t,\cdot), \quad \phi(0,0) = \alpha^0 \end{cases} \tag{1.2}$$

and

(i) $\phi_\xi(t,\xi) < 0$ in $\mathbb{R} \times \mathbb{R}$. Namely, $\phi(t,\cdot)$ is monotone decreasing with respect to the moving coordinate for each t.

(ii) There exist positive constants K and ν satisfying

$$\begin{cases} |\phi(t,\xi) - 1| + |\phi_\xi(t,\xi)| + |\phi_{\xi\xi}(t,\xi)| \le Ke^{\nu\xi}, \; \xi \le 0, \; t \in \mathbb{R}, \\ |\phi(t,\xi)| + |\phi_\xi(t,\xi)| + |\phi_{\xi\xi}(t,\xi)| \le Ke^{-\nu\xi}, \; \xi \ge 0, \; t \in \mathbb{R}. \end{cases} \tag{1.3}$$

That is, ϕ exponentially approaches its limits as $\xi \to \pm\infty$.

Moreover, they proved that such a time periodic traveling front is unique up to shifts in the spatial variable for any given unit vector $e \in \mathbb{S}^{N-1}$, and is asymptotically stable under front like planar initial perturbations. It is worth to point out that the level sets of such traveling fronts, which are orthogonal to the direction of propagation e, are parallel hyperplane. These fronts are invariant in