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ScienceDirect
J. Differential Equations 265 (2018) 2296–2339
Journal of Differential Equations
www.elsevier.com/locate/jde

Eventual smoothness of generalized solutions to a singular chemotaxis-Stokes system in 2D

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Received 12 May 2017; revised 12 April 2018

Abstract

We study the chemotaxis–fluid system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(\frac{n}{c}\nabla c), & x\in\Omega, \quad t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega, \quad t>0,\\ u_t + \ \nabla P \ = \Delta u + n\nabla\phi, & x\in\Omega, \quad t>0,\\ \qquad \nabla\cdot u \ = 0, & x\in\Omega, \quad t>0, \end{cases}$$

under homogeneous Neumann boundary conditions for n and c and homogeneous Dirichlet boundary conditions for u, where $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary and $\phi\in C^2(\bar{\Omega})$. From recent results it is known that for suitable regular initial data, the corresponding initial–boundary value problem possesses a global generalized solution. We will show that for small initial mass $\int_\Omega n_0$ these generalized solutions will eventually become classical solutions of the system and obey certain asymptotic properties.

Moreover, from the analysis of certain energy-type inequalities arising during the investigation of the eventual regularity, we will also derive a result on global existence of classical solutions under assumption of certain smallness conditions on the size of n_0 in $L^1(\Omega)$ and in $L\log L(\Omega)$, u_0 in $L^4(\Omega)$, and of ∇c_0 in $L^2(\Omega)$.

MSC: primary 35B65, 35B40; secondary 35K35, 35Q92, 92C17

Keywords: Chemotaxis; Stokes; Chemotaxis–fluid interaction; Global existence; Generalized solution; Eventual regularity; Stabilization

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https://doi.org/10.1016/j.jde.2018.04.035
0022-0396/

1. Introduction

Even among the smallest and most primitive organisms there are cases of complex and macroscopical collective behavior, for instance bacteria of species *E. coli* were confirmed to form migrating bands when subjected to a test environment featuring gradients of nutrient concentration ([1]). Following these experimental findings, chemotaxis systems with singular sensitivity of the form

$$\begin{cases} n_t = \Delta n - \nabla\cdot(\frac{n}{c}\nabla c), \\ c_t = \Delta c - nc, \end{cases} \tag{1.1}$$

were among the first phenomenological models proposed by Keller and Segel ([14]) to study these processes of chemotactic migration. Herein, n denotes the density of the bacteria which orient their movement towards increasing concentration c of a chemical substance which serves as their food source and is thereby consumed in the process. Singular chemotactic sensitivities of the type featured in (1.1) express the system assumption that the signal is perceived as described by the Weber–Fechner law ([11], [25]). An outstanding facet of this system, as already illustrated in [14], is the occurrence of wave-like solution behavior without any type of cell kinetics, which is known to be vital for such effects in standard reaction–diffusion equations. For studies on existence and stability properties of traveling wave solutions of (1.1) see [34,19,22] and references therein.

The results on global existence to systems of the form (1.1) are very sparse, with widely arbitrary initial data only being treated for the one-dimensional case ([30], [18]). In higher dimensions the results were constrained to the Cauchy problem for (1.1) in $\mathbb{R}^n$ with $n \in \{2, 3\}$, where smallness conditions on the initial data had to be imposed to show the existence of globally defined classical solutions ([35]). Only recently ([40]), so-called global generalized solutions to (1.1) were constructed in the two-dimensional case. The solutions are obtained through the study of a suitably chosen regularization guaranteeing that the regularized chemical concentration is strictly bounded away from zero for all times. These generalized solutions comply with the classical solution concept in the sense that generalized solutions which are sufficiently smooth also solve the system in the classical sense. In a sequel to the previously mentioned work the author furthermore proved that if the initial mass is small these generalized solutions eventually become classical solutions after some (possibly large) waiting time and that the solutions satisfy certain kind of asymptotic properties ([41]).

Eventual regularity and fluid interaction. Our interest slightly differing from the system proposed by Keller and Segel, where the model assumes no interaction between bacteria and surroundings, we will consider the case that the bacteria may be affected by their liquid environment. Here, we do not only assume that this interaction occurs by means of transport, but also in form of a feedback between the cells and the fluid velocity stemming from a buoyancy effect assumed in the model development featured in [31]. The experimental evidence reported in the latter reference suggests that the chemotactic motion inside the liquid can be substantially influenced by the feedback between cells and fluid, with turbulence emerging spontaneously in population of aerobic bacteria suspended in sessile drops of water. As a prototypical model for the description of this phenomenon a system of the form