ARTICLE IN PRESS YJDEQ:9267

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Journal of Differential Equations

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Positive solutions to a fractional equation with singular nonlinearity

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Received 7 June 2017

Abstract

In this paper, we study the positive solutions to the following singular and non local elliptic problem posed in a bounded and smooth domain $\Omega \subset \mathbb{R}^N$, $N > 2s$:

$$(P_\lambda)\begin{cases} (-\Delta)^s u = \lambda(K(x)u^{-\delta} + f(u)) \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u \equiv 0 \text{ in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Here $0 < s < 1$, $\delta > 0$, $\lambda > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive C^2 function. $K : \Omega \to \mathbb{R}^+$ is a Hölder continuous function in Ω which behave as $\text{dist}(x, \partial\Omega)^{-\beta}$ near the boundary with $0 \le \beta < 2s$.

First, for any $\delta > 0$ and for $\lambda >$ small enough, we prove the existence of solutions to (P_λ). Next, for a suitable range of values of δ, we show the existence of an unbounded connected branch of solutions to (P_λ) emanating from the trivial solution at $\lambda = 0$. For a certain class of nonlinearities f, we derive a global multiplicity result that extends results proved in [2]. To establish the results, we prove new properties which are of independent interest and deal with the behavior and Hölder regularity of solutions to (P_λ).

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https://doi.org/10.1016/j.jde.2018.03.023

0022-0396/

Please cite this article in press as: Adimurthi et al., Positive solutions to a fractional equation with singular nonlinearity, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.03.023

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N > 2s$, be a bounded domain with boundary of class $C^{1,1}$. In this work, we study solutions to the Problem (P_λ) above. Here $(-\Delta)^s$ is the fractional Laplace operator defined as

$$(-\Delta)^s u(x) = 2C(N,s)\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y$$

where P.V. denotes the Cauchy principal value and $C(N,s) = \pi^{-\frac{N}{2}} 2^{2s-1} s \dfrac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$, Γ being the Gamma function.

We assume that $0 < s < 1$, $\delta > 0$, $\lambda \geq 0$, $K \in C^{\nu}_{\text{loc}}(\Omega)$, $\nu \in (0,1)$, such that $\inf\limits_{\Omega} K > 0$ and satisfies for some $0 \leq \beta < 2s$ and $C_1, C_2 > 0$

$$C_1 d(x)^{-\beta} \leq K(x) \leq C_2 d(x)^{-\beta}, \quad \forall x \in \Omega \tag{1.1}$$

where $d(x) \stackrel{\text{def}}{=} \text{dist}(x, \partial\Omega)$.

Concerning f, we suppose the following conditions throughout the paper:

(f1) $f : [0,\infty) \to \mathbb{R}$ is a positive C^2 function with $f(0) = 0$;

(f2) The function $g_x : t \to \dfrac{K(x)}{t^\delta} + f(t)$ is strictly convex on $(0,\infty)$ for any $x \in \Omega$;

(f3) $\lim\limits_{t\to\infty} \dfrac{f(t)}{t} = \infty$ and there exists $C > 1$ such that $\liminf\limits_{t\to\infty} \dfrac{f'(t)t}{f(t)} \geq C$.

(f4) There exists $p \in \left(1, \dfrac{N+2s}{N-2s}\right)$ and $c > 0$ such that $\lim\limits_{t\to\infty} \dfrac{f(t)}{t^p} = c$.

(f5) There exists $q \in \left(1, \dfrac{N+2s}{N-2s}\right)$ such that $\dfrac{tf'(t)}{f(t)} \leq q$, for any $t > 0$.

The equation in (P_λ) has intrinsic mathematical interest since in the local setting $(s = 1)$ it appears in several physical models like non newtonian flows in porous media, heterogeneous catalysts (see references [15], [17], [18], [19] and the surveys [20] and [26]). The fractional laplacian case has been investigated more recently in [2] and [22] where the existence and multiplicity of solutions have been proved by variational methods of mountain pass type and the non smooth analysis theory. In these two papers, the authors restrict to the case $f(u) = u^p$, with $1 < p \leq \dfrac{N+2s}{N-2s}$, $\beta = 0$. Precisely, in [2], the subcritical case (i.e. $1 < p < \dfrac{N+2s}{N-2s}$) is considered. Existence of positive solutions are proved and a local multiplicity result is sketched for a certain range of δ. In [22], the critical case $p = \dfrac{N+2s}{N-2s}$ is dealt with and a global multiplicity result is proved for any $\delta > 0$. The solutions have the form $u_\lambda = \underline{u}_\lambda + v_\lambda$ where $v_\lambda \in \tilde{H}^s(\Omega)$ and $\underline{u}_\lambda$ is the solution to the "pure singular" problem (see (P_s) below), i.e. $\underline{u}_\lambda$ satisfies:

$$(P_s) \begin{cases} (-\Delta)^s \underline{u}_\lambda = \lambda K(x) \underline{u}_\lambda^{-\delta} \text{ in } \Omega, \\ \underline{u}_\lambda > 0 \text{ in } \Omega, \\ \underline{u}_\lambda \equiv 0 \text{ in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

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