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Finite energy of generalized suitable weak solutions to the Navier–Stokes equations and Liouville-type theorems in two dimensional domains

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Abstract

Introducing a new notion of *generalized suitable weak solutions*, we first prove validity of the energy inequality for such a class of weak solutions to the Navier–Stokes equations in the whole space $\mathbb{R}^n$. Although we need certain growth condition on the pressure, we may treat the class even with infinite energy quantity except for the initial velocity. We next handle the equation for vorticity in 2D unbounded domains. Under a certain condition on the asymptotic behavior at infinity, we prove that the vorticity and its gradient of solutions are both globally square integrable. As their applications, Loiuville-type theorems are obtained.

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Keywords: Navier–Stokes equations; Energy inequalities; Liouville-type theorems

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1. Introduction

We consider the Cauchy problem for the Navier–Stokes equations

$$\begin{cases} v_t - \Delta v + (v\cdot\nabla)v + \nabla p = 0, & (x,t)\in\mathbb{R}^n\times(0,T), \\ \operatorname{div} v = 0, & (x,t)\in\mathbb{R}^n\times(0,T), \\ v(x,0) = v_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $v = v(x,t) = (v^1(x,t),\dots,v^n(x,t))$ and $p = p(x,t)$ denote the velocity and the pressure, respectively, while $v_0(x) = (v_0^1(x),\dots,v_0^n(x))$ stands for the given initial velocity. Let the initial data v_0 belong to $L^2_\sigma(\mathbb{R}^n)$, which is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n)$, compactly supported C^∞-solenoidal vector functions, with respect to the L^2-norm. We recall that a measurable function v on $\mathbb{R}^n\times(0,T)$ is a weak solution of the Leray–Hopf class to (1.1) if $v\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^n))\cap L^2_{loc}([0,T);H^1_\sigma(\mathbb{R}^n))$ and if v satisfies (1.1) in the sense that

$$\int_0^T \{-(v,\partial_\tau\Phi) + (\nabla v,\nabla\Phi) + (v\cdot\nabla v,\Phi)\}\,d\tau = (v_0,\Phi(0))$$

holds for all $\Phi\in H^1_0([0,T);H^1_\sigma(\mathbb{R}^n)\cap L^n(\mathbb{R}^n))$. For every weak solution $v(t)$ of the Leray–Hopf class to (1.1), it is shown by Prodi [18] and Serrin [20] that, after a redefinition of its value of $v(t)$ on a set of measure zero in the time interval $[0,T]$, $v(\cdot,t)$ is continuous for t in the weak topology of $L^2_\sigma(\mathbb{R}^n)$. See also Masuda [17, Proposition 2].

Serrin [20] proved that if v is a weak solution of the Leray–Hopf class to (1.1) and if $v\in L^s(0,T;L^q(\mathbb{R}^n))$ for $\frac{3}{q}+\frac{2}{s}\le 1$ with some $q>3, s>2$, then the energy identity

$$\|v(t)\|^2_{L^2} + 2\int_0^t \|\nabla v(\tau)\|^2_{L^2}d\tau = \|v_0\|^2_{L^2} \quad (0\le t<T) \tag{1.2}$$

is valid. Shinbrot [21] also proved that the same conclusion holds under another assumption for some $s>1, q\ge 4$ such that $\frac{2}{q}+\frac{2}{s}\le 1$. Taniuchi [23] further extended these results to

$$\begin{aligned} &\frac{2}{q}+\frac{2}{s}\le 1, \quad \frac{3}{q}+\frac{1}{s}\le 1 \quad (n=3), \\ &\frac{2}{q}+\frac{2}{s}\le 1, \quad q\ge 4 \quad (n\ge 4). \end{aligned}$$

Recently, Farwig–Taniuchi [6] obtained a new class by means of domains of fractional powers of the Stokes operator in general unbounded domains in $\mathbb{R}^3$.

In this paper, we give a new condition which ensures the energy inequality, and as its application, several Liouville-type theorems are established. Let us first introduce our definition of a generalized suitable weak solution.