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Integrability of scalar curvature and normal metric on conformally flat manifolds

Shengwen Wang, Yi Wang [*,1]

Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, United States of America

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Abstract

On a manifold $(\mathbb{R}^n, e^{2u}|dx|^2)$, we say u is normal if the Q-curvature equation that u satisfies $(-\Delta)^{\frac{n}{2}}u = Q_g e^{nu}$ can be written as the integral form $u(x) = \frac{1}{c_n}\int_{\mathbb{R}^n} \log\frac{|y|}{|x-y|} Q_g(y)e^{nu(y)}dy + C$. In this paper, we show that the integrability assumption on the negative part of the scalar curvature implies the metric is normal. As an application, we prove a bi-Lipschitz equivalence theorem for conformally flat metrics.

MSC: primary 53A30; secondary 53C21

1. Introduction

Let (M^n, g) be a smooth Riemannian manifold. The Q-curvature arises naturally as a conformal invariant associated to the Paneitz operator. When $n = 4$, the Paneitz operator is defined as:

$$P_g = \Delta_g^2 + \delta(\frac{2}{3}R_g\, g - 2\,\mathrm{Ric}_g)d,$$

* Corresponding author.
E-mail addresses: swang@math.jhu.edu (S. Wang), ywang@math.jhu.edu (Y. Wang).
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where δ is the divergence operator, d is the differential operator, R is the scalar curvature of g, and Ric is the Ricci curvature tensor. The Branson's Q-curvature [3] is defined as

$$Q_g = \frac{1}{12}\left\{-\Delta_g R_g + \frac{1}{4}R_g^2 - 3|E_g|^2\right\},$$

where E_g is the traceless part of Ric_g, and $|\cdot|$ is the point-wise norm taken with respect to the metric g. Suppose $u \in C^\infty(M)$ is a smooth function. Under the conformal change $g = e^{2u}g_0$, the Paneitz operator transforms by $P_g = e^{-4u}P_{g_0}$, and Q_g satisfies the fourth order equation

$$P_{g_0}u + 2Q_{g_0} = 2Q_g e^{4u}. \tag{1.1}$$

This is analogous to the transformation law satisfied by the Laplacian operator $-\Delta_g$ and the Gaussian curvature K_g on surfaces,

$$-\Delta_{g_0}u + K_{g_0} = K_g e^{2u}. \tag{1.2}$$

When the background metric g_0 is the flat metric $|dx|^2$, the transformation law (1.1) that the Q-curvature satisfies becomes

$$\Delta_{g_0}^2 u = 2Q_g e^{4u}. \tag{1.3}$$

The invariance of the integration of the Q-curvature in dimension 4 is due to the Gauss–Bonnet–Chern formula for a closed manifold M:

$$\chi(M) = \frac{1}{4\pi^2}\int_M \left(\frac{|W_g|^2}{8} + Q_g\right) dv_g, \tag{1.4}$$

where W_g denotes the Weyl tensor. For complete manifolds with conformally flat ends, the work of Chang, Qing, and Yang [4] proves the formula between the asymptotic isoperimetric ratio and the integration of the Q-curvature.

In Chang, Qing and Yang's work [4,5], they used an important notion "normal metric" on conformally flat manifolds to prove the formula of the asymptotic isoperimetric ratio. Normal metric was first introduced by Huber [9], and later used by Finn [7] and [8]. Huber proved that in dimension two, for a surface with finite total Gauss curvature, the metric is always normal. In [4]'s work, it is a key observation that if the scalar curvature at infinity is nonnegative, then the metric is normal. The proof mostly uses maximum principle and properties of harmonic functions. In [11] and [12], being a normal metric is an important condition to study geometric implications of the Q-curvature on conformally flat manifolds. In this paper, we generalize this result. We show that if the negative part of the scalar curvature is integrable, then the metric is normal. The main result is the following theorem.

Theorem 1.1. *Let $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ be a noncompact complete conformally flat metric of even dimension, satisfying*

$$\int_{M^n} |Q_g| dv_g < \infty, \tag{1.5}$$