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An energy method for rough partial differential equations ☆

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Abstract

We present a well-posedness and stability result for a class of nondegenerate linear parabolic equations driven by geometric rough paths. More precisely, we introduce a notion of weak solution that satisfies an intrinsic formulation of the equation in a suitable Sobolev space of negative order. Weak solutions are then shown to satisfy the corresponding energy estimates which are deduced directly from the equation. Existence is obtained by showing compactness of a suitable sequence of approximate solutions whereas uniqueness relies on a doubling of variables argument and a careful analysis of the passage to the diagonal. Our result is optimal in the sense that the assumptions on the deterministic part of the equation as well as the initial condition are the same as in the classical PDEs theory.

Keywords: Rough paths; Rough PDEs; Energy method; Weak solutions

1. Introduction

The so-called variational approach, also known as the energy method, belongs among the most versatile tools in the theory of partial differential equations (PDEs). It is especially useful for nonlinear problems with complicated structure which do not permit the use of (semi-) linear methods such as semigroup arguments, e.g. systems of conservation laws or equations appearing

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in fluid dynamics. In such cases, solutions are often known or expected to develop singularities in finite time. Therefore, weak (or variational) solutions which can accommodate these singularities provide a suitable framework for studying the behavior of the system in the long run. But even for linear or semi-linear problems, weak solutions are the natural notion of solution in cases where a corresponding mild formulation is not available, for instance due to low regularity of coefficients.

The construction of weak solutions via the energy method relies on basic a priori estimates which can be directly deduced from the equation at hand by considering a suitable test function. The equation is then satisfied in a weak sense, that is, as an equality in certain space of distributions. Within this framework, existence and uniqueness are usually established by separate arguments. The proof of existence often uses compactness of a sequence of approximate solutions. Uniqueness for weak solutions is much more delicate and in some cases even not known. Let us for instance mention problems appearing in fluid dynamics where the questions of uniqueness and regularity of weak solutions remain largely open.

It has been long recognized that addition of stochastic terms to the basic governing equations can be used to model an intrinsic presence of randomness as well as to account for other numerical, empirical or physical uncertainties. Consequently, the field of stochastic partial differential equations massively gained importance over the past decades. It relies on the (martingale based) stochastic Itô integration theory, which gave a probabilistic meaning to problems that are analytically ill-posed due to the low regularity of trajectories of the driving stochastic processes. Nevertheless, the drawback appearing already in the context of stochastic differential equations (SDEs) is that the solution map which assigns a trajectory of the solution to a trajectory of the driving signal, known as the Itô map, is measurable but in general lacks continuity. This loss of robustness has obvious negative consequences, for instance when dealing with numerical approximations or in filtering theory.

The theory of rough paths introduced by Lyons [22] fully overcame the gap between ordinary and stochastic differential equations and allowed for a pathwise analysis of SDEs. The highly nontrivial step is lifting the irregular noise to a bigger space in a robust way such that solutions to SDEs depend continuously on this lifted noise. More precisely, Lyons singled out the appropriate topology on the space of rough paths which renders the corresponding Itô–Lyons solution map continuous as a function of a suitably enhanced driving path. As one of the striking consequences, one can allow initial conditions as well as the coefficients of the equation to be random, even dependent on the entire future of the driving signals – as opposed to the "arrow of time" and the associated need for adaptedness within Itô's theory. In addition, using the rough path theory one can consider drivers beyond the martingale world such as general Gaussian or Markov processes, in contrast to Itô's theory where only semimartingales may be considered.

The rough path theory can be naturally formulated also in infinite dimensions to analyze ODEs in Banach spaces. This generalization is, however, not appropriate for the understanding of rough PDEs. This is due to two basic facts. First, the notion of rough path encodes in a fundamental way the nonlinear effects of time varying signals without any possibility of including signals depending in an irregular way on more parameters. Second, in an infinite dimensional setting the action of a signal (even finite dimensional) is typically described by differential or more generally unbounded operators. Due to these difficulties, attempts at application of the rough path theory in the study rough PDEs have been limited. Namely, it was necessary to avoid unbounded operators by working with mild formulations or Feynman–Kac formulas or transforming the equation in order to absorb the rough dependence into better understood objects such as flow of characteristic curves.

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