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Metastability of the Cahn–Hilliard equation in one space dimension

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Abstract

We establish metastability of the one-dimensional Cahn–Hilliard equation for initial data that is order-one in energy and order-one in $\dot{H}^{-1}$ away from a point on the so-called slow manifold with N well-separated layers. Specifically, we show that, for such initial data on a system of lengthscale Λ, there are three phases of evolution: (1) the solution is drawn after a time of order Λ^2 into an algebraically small neighborhood of the N-layer branch of the slow manifold, (2) the solution is drawn after a time of order Λ^3 into an exponentially small neighborhood of the N-layer branch of the slow manifold, (3) the solution is trapped for an exponentially long time exponentially close to the N-layer branch of the slow manifold. The timescale in phase (3) is obtained with the sharp constant in the exponential.

Keywords: Dynamic metastability; Gradient flow; Nonlinear PDE; Energy methods; Coarsening rates

1. Introduction

Local energy minimizers are the stable states of a gradient flow: Solutions started at the minimizers are in equilibrium, and solutions started nearby relax towards this equilibrium state. There are however physical systems that exhibit *metastability*: Solutions appear to be stationary, but are in fact far from any stable state and evolving on an extremely long timescale. (This behavior is

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called dynamic metastability to distinguish it from the noise-induced metastability of stochastic systems.)

Two fundamental examples displaying dynamic metastability are the one-dimensional Allen–Cahn equation

$$u_t = u_{xx} - G'(u), \qquad x \in \left(-\tfrac{\Lambda}{2}, \tfrac{\Lambda}{2}\right), \ t > 0, \tag{1.1}$$

and the one-dimensional Cahn–Hilliard equation

$$u_t = -(u_{xx} - G'(u))_{xx}, \qquad x \in \left(-\tfrac{\Lambda}{2}, \tfrac{\Lambda}{2}\right), \ t > 0, \tag{1.2}$$

subject to suitable boundary conditions. Here $G(u)$ is a double-well potential with nondegenerate minima at ± 1 (cf. Remark 1.1). Both (1.1) and (1.2) are often studied with a small parameter ε appearing in the equation. This is equivalent via rescaling to studying (1.1) and (1.2) on an interval with $\Lambda \gg 1$, which is the setting that we will consider in this paper.

Both equations represent phenomenological models for the coexistence of two "pure" phases ± 1, and the value of the order parameter u indicates the proportion of each phase. An important physical and mathematical property of the Cahn–Hilliard equation (under appropriate boundary conditions) is that it preserves the mean:

$$\fint_{\left(-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right)} u(x,t)\,\mathrm{d}x = \fint_{\left(-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right)} u(x,0)\,\mathrm{d}x =: m \qquad \text{for all } t > 0.$$

Equations (1.1) and (1.2) can be derived from the scalar Ginzburg–Landau energy

$$E(u) := \int_{\left(-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right)} \frac{1}{2}u_x^2 + G(u)\,\mathrm{d}x. \tag{1.3}$$

Equations (1.1) and (1.2) are the gradient flows of (1.3) with respect to the L^2 metric and the $\dot{H}^{-1}$ metric, respectively.

The metastability of equation (1.1) has been well-analyzed; see [6–8,10,12] and the discussion in subsection 1.2. The generic picture can be described in the following way. For initial data with large regions of positive phase interspersed with large regions of negative phase, it is observed that the solution quickly settles down to a configuration with large regions of $u \approx \pm 1$ that are connected by so-called transition layers. These transition layers are well-approximated by energy minimizers on $\mathbb{R}$ connecting ± 1 boundary conditions at infinity. Subsequently the solution appears almost stationary until a time that is of exponential order with respect to the distance between zeros. Roughly speaking, the collection of states with N optimal transition layers connecting ± 1 forms a slow motion manifold for the system: The system quickly relaxes to the slow manifold and then evolves slowly along it. However the solution is far from the final state. Indeed, suppose that the two closest transition layers are a distance ℓ apart. After a time that is exponentially long with respect to ℓ, these two layers come together and collapse, reducing the energy and producing a state with $N-2$ transitions. Again the solution remains metastable for an exponentially long time until the next two layers come together, and so on, until the last pair

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