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J. Differential Equations 265 (2018) 1576–1595

Journal of Differential Equations

www.elsevier.com/locate/jde

# Radial solutions of a fourth order Hamiltonian stationary equation ☆

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Received 16 July 2017
Available online 24 April 2018

**Abstract**

We consider smooth radial solutions to the Hamiltonian stationary equation which are defined away from the origin. We show that in dimension two all radial solutions on unbounded domains must be special Lagrangian. In contrast, for all higher dimensions there exist non-special Lagrangian radial solutions over unbounded domains; moreover, near the origin, the gradient graph of such a solution is continuous if and only if the graph is special Lagrangian.


## 1. Introduction

An important class of Lagrangian submanifolds in a symplectic manifold is the so-called Hamiltonian stationary Lagrangian submanifolds, which are Lagrangian submanifolds and are critical points of the volume functional under Hamiltonian variations. A well-known subset of this class consists of the special Lagrangian submanifolds. These are Lagrangian and critical for the volume functional for all variations; so they are minimal Lagrangian submanifolds. The special Lagrangians form one of the most distinguished classes in calibrated geometry, and they play a unique role in string theory.

☆ The first author was supported in by NSERC Discovery Grant 03709. The second author was partially supported by NSF Grant DMS-1438359.
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https://doi.org/10.1016/j.jde.2018.04.011
0022-0396/

For a fixed bounded domain $\Omega \subset \mathbb{R}^n$, let $u : \Omega \to \mathbb{R}$ be a smooth function. The gradient graph $\Gamma_u = \{(x, Du(x)) : x \in \Omega\}$ is a Lagrangian $n$-dimensional submanifold in $\mathbb{C}^n$, with respect to the complex structure $J$ defined by $z_j = x_j + iy_j$ for $j = 1, \cdots, n$. The volume of $\Gamma_u$ is given by

$$F_\Omega(u) = \int_\Omega \sqrt{\det\left(I + \left(D^2u\right)^T D^2u\right)}dx.$$

A smooth function $u$ is critical for the volume functional $F_\Omega(u)$ under compactly supported variations of the scalar function if and only if $u$ satisfies the equation

$$\Delta_g \theta = 0 \tag{1.1}$$

where $\Delta_g$ is the Laplace–Beltrami operator on $\Gamma_u$ for the induced metric $g$ from the Euclidean metric on $\mathbb{R}^{2n}$, (cf. [8], [10, Proposition 2.2]). Here, the Lagrangian phase function is defined by

$$\theta = \operatorname{Im}\log\det\left(I_n + \sqrt{-1}D^2u\right)$$

or equivalently,

$$\theta = \sum_{i=1}^{n} \arctan \lambda_i \tag{1.2}$$

for $\lambda_i$ the eigenvalues of $D^2u$. The mean curvature vector along $\Gamma_u$ can be written

$$\vec{H} = -J\nabla\theta$$

where $\nabla$ is the gradient operator of $\Gamma_u$ for the metric $g$ (cf. [6, 2.19]).

The gradient graph of $u$ which solves (1.1) is called Hamiltonian stationary. A Hamiltonian stationary gradient graph $\Gamma_u$ is a critical point of the volume functional $F_\Omega(\cdot)$ under Hamiltonian deformations, that is, those generated by $J\nabla\eta$ for some smooth compactly supported function $\eta$. On the other hand, recall that if $u$ satisfies the special Lagrangian equation [6]

$$\nabla\theta = 0 \tag{1.3}$$

i.e. $\vec{H} \equiv 0$, then the surface is critical for the volume functional under *all* compactly supported variations of the surface $\Gamma_u$. In terms of the potential function $u$, the Hamiltonian stationary equation is of fourth order and the special Lagrangian equation is of second order, both are elliptic. There are Hamiltonian stationary but not special Lagrangian surfaces even when $n = 2$, and this causes serious problems for constructing special Lagrangian surfaces (see [10]).

In this paper, we consider radial solutions of the Hamiltonian stationary equation (1.1) on a domain which may not contain the origin. Our first observation is that the fourth order Hamiltonian stationary equation necessarily reduces to the second order special Lagrangian equation for radial solutions defined on any *unbounded* domain in $\mathbb{R}^2$; however, this is not the case for $n > 2$.