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Minimization of quotients with variable exponents

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Abstract

Let Ω be a bounded domain of $\mathbb{R}^N$, $p \in C^1(\overline{\Omega})$, $q \in C(\overline{\Omega})$ and $l, j \in \mathbb{N}$. We describe the asymptotic behavior of the minimizers of the Rayleigh quotient $\frac{\|\nabla u\|_{lp(x)}}{\|u\|_{jq(x)}}$, first when $j \to \infty$ and after when $l \to \infty$.

MSC: 35B40; 35J60; 35P30

Keywords: Asymptotic behavior; Infinity Laplacian; Variable exponents

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, and consider the Rayleigh quotient

$$\frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}},$$

associated with the immersion of the Sobolev space $W_0^{1,p(x)}(\Omega)$ into the Lebesgue space $L^{q(x)}(\Omega)$, where the variable exponents satisfy

$$1 < \inf_\Omega p(x) \leq \sup_\Omega p(x) < \infty$$

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and

$$1 < q(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N \\ \infty & \text{if} \quad p(x) \geq N. \end{cases}$$

In this paper we study the behavior of the least Rayleigh quotients when the functions $p(x)$ and $q(x)$ become arbitrarily large. Our script is based on the paper [8], where these functions are constants. Thus, in order to overcome the difficulties imposed by the fact that the exponents depend on x, we adapt arguments developed by Franzina and Lindqvist in [18], where $p(x) = q(x)$. Actually, our results in the present paper generalize those of [8] for variable exponents and complement the approach of [18].

In [8], Ercole and Pereira first studied the behavior, when $q \to \infty$, of the positive minimizers w_q corresponding to

$$\lambda_q := \min\left\{\|\nabla u\|_{L^p(\Omega)} : u \in W_0^{1,p}(\Omega) \quad \text{in} \quad \|u\|_{L^q(\Omega)} = 1\right\},$$

for a fixed $p > N$. They obtained a function $u_p \in W_0^{1,p}(\Omega)$ as the uniform limit in $\overline{\Omega}$ of a sequence w_{q_n}, with $q_n \to \infty$. Such a function is positive in Ω, assumes the maximum value 1 at a unique point x_p and satisfies

$$\begin{cases} -\Delta_p u = \Lambda_p \delta_{x_p} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\Lambda_p := \min\left\{\|\nabla u\|_{L^p(\Omega)} : u \in W_0^{1,p}(\Omega) \quad \text{in} \quad \|u\|_{L^\infty(\Omega)} = 1\right\}$$

and δ_{x_p} denotes the Dirac delta distribution concentrated at x_p. In the sequence, they determined the behavior of the pair (Λ_p, u_p), as $p \to \infty$. In fact, they proved that

$$\lim_{p\to\infty} \Lambda_p = \Lambda_\infty := \inf_{0 \not\equiv v \in W_0^{1,\infty}(\Omega)} \frac{\|\nabla v\|_\infty}{\|v\|_\infty}$$

and that there exist a sequence $p_n \to \infty$, a point $x_* \in \Omega$ and a function $u_\infty \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that: $x_{p_n} \to x_*$, $\|d\|_\infty = d(x_*)$, where d is the distance function to the boundary, $u_\infty \leq \frac{d}{\|d\|_\infty}$ and $u_{p_n} \to u_\infty$ uniformly in $\overline{\Omega}$. Moreover, they showed that: u_∞ is also a minimizer of Λ_∞, assumes its maximum value 1 only at x_* and satisfies

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \backslash \{x_*\} \\ u = \frac{d}{\|d\|_\infty} & \text{on } \partial\left(\Omega \backslash \{x_*\}\right) = \{x_*\} \cup \partial\Omega \end{cases}$$

in the viscosity sense.

In [18], Franzina and Lindqvist determined the exact asymptotic behavior, as $j \to \infty$, of both the minimum $\Lambda_{jp(x)}$ of the quotients $\frac{\|\nabla u\|_{jp(x)}}{\|u\|_{jp(x)}}$ and its respective $jp(x)$-normalized minimizer u_j. They proved that

$$\lim_{j\to\infty} \Lambda_{jp(x)} = \Lambda_\infty$$